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*Research Article*

# Solutions of Nonlinear Higher-Order Boundary Value Problems by using Differential Transformation Method and Adomian Decomposition Method

**[1]Muhammad Nadeem, [2]Afifa Maqsood, [3]Muhammad Moaz Safer, [1,4]Hira Fatima, [1,5]Abdul Raheem and *[6]Tazeem Fatima**

[1,4,5]Department of Mathematics, Lahore Leads University, Pakistan
[2]Department of Mathematics & Statistics, Bahauddin Zakariya University, Pakistan
[3]Department of Mathematics & Statistics, University of Agriculture Faisalabad, Pakistan
[*6]Department of Mathematics, Government College University Faisalabad, Pakistan

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### *Abstract*

*Comparison of Numerical Techniques for the solutions of Nonlinear Singular Differential Equations is a very important topic in applied mathematics. In this work, we present a powerful method for the numerical solution of non-linear singular initial value problems. We calculate approximate solutions for nonlinear singular initial value problems by using modified Laplace decomposition method and also by the use of modified adomian decomposition method. In this work, we present a powerful method for the numerical solution of non-linear singular boundary value problems, namely the advanced Adomian decomposition method which is the modification of adomian decomposition method. In this method, we use all the boundary conditions to obtain the coefficient of the approximate series solution. Moreover, convergence analysis and an error bound for the approximate solution are discussed. This method overcomes the singular behaviour of problems and illustrates the approximations of high precision with a large effective region of convergence. To prove the robustness and effectiveness of the proposed method, various examples are considered and the obtained results are compared with the other existing methods. This method overcomes the singular behaviour of problems and illustrates the approximations of high precision with a large effective region of convergence. To prove the effectiveness of the proposed method, various examples are considered and the obtained results are compared with the other existing methods. For the accuracy of results we have calculated the solution up to more than three places.it overcome the singular behaviour of problems and also find the approximation with high accuracy by using these methods. We will find the effectiveness or pact of proposed methods. Different example is taken and the obtained results are compared with the previous researched. Error estimation and convergence order for the presented method are also discussed. Our numerical experiments validate our theoretical findings and demonstrate the performance of the algorithm in terms of simplicity, accuracy, and efficiency.*

***Keywords:*** *Numerical Techniques, Adomian Decomposition Method, Differential Transformation Method*

## 1. Introduction

### The Differential Transformation Method (DTM)

A point-cantered definition of a k th function's -order differential transformation y(x) = f(x)

$x=x_0$as:

$$\mathrm{Y(k)} = \frac{1}{\mathrm{k!}}\left[\frac{\mathrm{d^k y(x)}}{\mathrm{dx^k}}\right] \tag{2.1}$$

*Corresponding author's ORCID ID: 0000-0002-0921-086X
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When k is a member of the K-domain, a collection of non-negative integers. The differential transformations can be used to express the function y(x), Y(k) in this way as:

$$y(x) = \sum_{k=0}^{\infty}\left[\frac{x-x_0}{k!}\right]Y(k) \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$\mathrm{y(x)} = \sum_{\mathrm{k}=0}^{\infty}(\mathrm{x}-\mathrm{x_0})\frac{1}{\mathrm{k!}}\left[\frac{\mathrm{d^k y(x)}}{\mathrm{dx^k}}\right]_{\mathrm{x=x_0}} \tag{2.3}$$

Which is basically the series for Taylor y(x) about $x = x_0$

One can derive certain principles of transformational operations from the fundamental definition of differential transforms (DT), The following is a list of some of those.

(1) : if z(x) =u(x) then, Z(k) =U(k) $\pm V(k)$.
(2) :if z(x)=$\propto u(x) then, Z(k) = \propto U(k). here \propto is\ a\ constant$.
(3) : if z(x)=$\frac{du(x)}{dx} then, Z(k) = (k+1)U(K+1)$.
(4) : if z(x)=$\frac{d^2u(x)}{dx^2} then, Z(k) = (k+1)(k+2)U(K+2)$.
(5) : if z(x)=$\frac{d^nu(x)}{dx^m} then, Z(k) = (k+1)(k+2)\ldots\ldots(k+m)U(k+m)$.
(6) : if z(x) =u(x)v(x)then, Z(k) =$\sum_{i=0}^{k} V(1)U(k-1)$.
(7) :if z(x) =$x^m then, z(k) = \Im(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n, \end{cases}$
(8) : if z(x) = exp $(\lambda x) then, Z(k) = \frac{\lambda^k}{k!}$.
(9) : if z(x) =(1+x)$^m$ then, Z(k) =$\frac{m(m-1)-(m-k+1)}{k!}$
(10) : if z(x) $= \sin(\omega x+\alpha)\ the, Z(k) = \frac{\omega^k}{k!}\sin\left(\frac{\pi k}{2}+\alpha\right)$.
(11) : if z(x) =$\cos(\omega x+\alpha)\ then, Z(k) = \frac{\omega^k}{k!}\cos\left(\frac{\pi k}{2}+\alpha\right)$.

HOBPs

Problems with even-order boundary values

Let the special (2rn) –order boundary value problem of the form

$$y^{2m}(x) = f(x,y), \qquad 0 < x < b, \tag{2.4}$$

*with boundary conditions*

$$y^{2j}(0) = \alpha_{2j}; j = 0,1,2,,,,.(m-1) \tag{2.5}$$

$$y^{2j}(0) = \beta 2J; = 0,1,2,,,,.(m-1) \tag{2.6}$$

(i) Odd-order boundary value problems

Learn about the form's special (2m+1) order BVP.

$$y^{2m+1}(x) = f(x,y), \quad 0 < x < b, \tag{2.7}$$

*with boundary conditions*

$$y^{(2J+1)}(0) = \gamma_{2j+1}; \quad j = 0,1,2,\ldots\ldots.m, \tag{2.8}$$

$$y^{(2J+1)}(b) = \gamma_{2j+1}; \quad j = 0,1,2,\ldots\ldots.m, \tag{2.9}$$

It's interesting to notice that for x∈[0,b]and _2j and _2j,j=0,1,2, y(x) and f(x,y) are assumed to be genuine and as many times differentiable as necessary. The criteria _2j,j=0,1,2,...,(m-1)describe the border even-order derivatives with x=0, and (m-1)describe the real finite constants of the even order while $\gamma_{2j+1} and\ \gamma_{2j+1\ j} 0,1,2\ldots.m\ describe$ the odd-order are real finite constants and the conditions $\gamma_{2j+1}j = 0,1,2,\ldots\ldots m$ describe the odd-order derivatives at the boundary x=b.

Although there are no numerical methods for addressing HOBVPs of higher order there, the book by Agarwal [1] has theorems that describe the prerequisites for the existence and uniqueness solution of such problems.

**General Differential Transformation**

The following generalised theorem is now proven.

The generic differential transformation for nth-order, nonlinear BVPs, Theorem 3.1

$y^x(x) = e^{-1}y^m(x), is\ given\ by$

$$Y(n+k) = \frac{k!}{(n+k)!}\left[\sum_{k_m=0}^{k} ⬚ \sum_{k_m=1-0}^{k_m} ⬚ \sum_{k_1=0}^{k_2}\left(-\frac{(-1)_1^k}{k_1}\right)\left(\left[\prod_{i=2}^{m} Y(k_i - k_{i-1})\right]Y(k-k_m)\right]\right.$$

**Proof**

We use the induction approach to prove this theorem. First, we establish that the differential equation can be generalised.$e^{-1}y^m(x). Let\ m = 2, then,$

$y^n(x) = e^{-x}y^m(x)$.

For k≥ 1, we have.

$$y^{n+k}(x) = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!(-1)_1^k e^{-x}}{k!(k_2-k_1)!(k-k_2)!}y^{(k_2-k_1)}(x)y^{(k-k_2)}(x),$$

$$y^{n+k}(x)|\_x = 0 = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!(-1)_1^k e^{-x}}{k!(k_2-k_1)!(k-k_2)!}y^{(k_2-k_1)}(x)y^{(k-k_2)}(x),$$

By definition, we have.

$$(n+k)!\,Y(n+k) = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!(-1)^{k_1}}{k_1!(k_2-k_1)!(k-k_2)!}(k_2-k_1)!\,Y(k_2-k_1)(k-k_2)!\,Y(k-k_2,$$

$$(n+k)!\,Y(n+k) = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!(-1)^{k_1}}{k_1!}Y(k_2-k_1)Y(k-k_2)$$

$$Y(n+k) = \frac{k!}{(n+k)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1!}Y(k_2-k_1)Y(k-k_2)$$

From (2.4) , for m=2, we have

$$Y(n+k) = \frac{k!}{(n+k)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1!}Y(k_2-k_1)Y(k-k_2)$$

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This suggests that for m=2, the theorem is true. Assume for the moment that the DT is as follows for m=p:

$$Y(n+k) = \frac{k!}{(n+k)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\ldots\ldots\sum_{k_1=0}^{k_2}\left(\frac{(-1)^{k_1}}{k_1!}\right)\left(\left[\prod_{i=0}^{p} Y\ (k_i - k_{i-1})\right]Y(k-k_p)\right]$$

$$Y(n+k) = \frac{k!}{(n+k)!}\times \left[\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\ldots\ldots\sum_{k_1=0}^{k_2}\left(\frac{(-1)^{k_1}}{k_1!}\right)\left(Y(k_2-k_1)Y(k_3-k_2)\ldots\ldots\ldots\ldots.(k_p-k_{p-1})Y(k-k_p)\right)\right]$$

Note that m=p we have.

$y^{(n)}(x) = e^{-x}y^p(x)$.

For $k \geq 1, we\ have$

$$y^{(n+k)}(x)|_{x=0} = \sum_{k_p=0}^{k}\sum_{k_{p-1}=0}^{k_2}\ldots\ldots\sum_{k_1=0}^{k_2}\frac{k!(-1)^{k_1}y^{(k_2-k_1)}(x)y^{(k_3-k_2)}(x)y^{(k-k_p)}(x)}{k_1!(k_2-k_1)!(k_3-k_2)!\ldots\ldots\ldots(k_p-k_{p-1})!(k-k_p)!}$$

Thus, for m=p+1, we have

$y^{(n)}(x) = e^{-x}y^{p}(x)y(x)$

For $k \geq 1$, we have

$y^{(n+k)}(x)|_{x=0} =$

$\sum_{k_p=0}^{k}\sum_{k_{p-1}=0}^{k_2} \ldots \ldots \sum_{k_1=0}^{k_2} \frac{k!(-1)^{k_1}y^{(k_2-k_1)}(x)y^{(k_3-k_2)}(x)\ldots\ldots y^{(k_{p+1}-k_p)}(x)y^{k-k_{p+1}(x)}}{k_1!(k_2-k_1)!(k_3-k_2)!\ldots\ldots\ldots(k_p-k_{p-1})!(k-k_p)!}$

By definitions, we have

$(n+k)!\,(n+k) = \sum_{k_{p-1}=0}^{k_2} \ldots \ldots \sum_{k_1=0}^{k_2} \frac{k!(-1)^{k_1}(k_2-k_1)!Y(k_2-k_1)\ldots(k-k_{p+1})!Y(k-k_{p+1})}{k_1!(k_2-k_1)!(k_3-k_2)!\ldots(k_{p+1}-k)!(k-k_{p+1})!}$

By definition, we have

$(n+k)!\,Y(n+k) = \sum_{k_{p-1}=0}^{k_2} \ldots \ldots \sum_{k_1=0}^{k_2} \frac{k!(-1)^{k_1}(k_2-k_1)!Y(k_2-k_1)\ldots(k-k_{p+1})!Y(k-k_{p+1})}{k_1!(k_2-k_1)!\ldots.(k-k_{p+1})!}$

$Y(n+k) =$

$\frac{k!}{(n+k)!}\sum_{k_{p-1}=0}^{k_2}\sum_{k_p=0}^{k_{p+1}} \ldots \ldots \sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1}Y(k_2 - k_1)!\,Y(k_3-k_2)\ldots.(k_{p+1}-k_p)Y(k-k_{p+1}).$

Mathematical Problems in Engineering From (3.1) for m=p+1, we have

$Y(n+k) =$

$\frac{k!}{(n+k)!}\sum_{k_{p-1}=0}^{k_2}\sum_{k_p=0}^{k_{p+1}} \ldots \ldots \ldots \sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1}(\prod_{i=2}^{p+1}Y(k_i - k_{i-1}]Y(k-k_{p+1})]$ ,

$Y(n+k) =$

$\frac{k!}{(n+k)!}\sum_{k_{p-1}=0}^{k_2}\sum_{k_p=0}^{k_{p+1}} \ldots \ldots \ldots \sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1}.Y(k_2 - k_1)Y(k_3-k_2)\ldots(k_{p+1}-k_p)Y(k-k_{p+1}).$

This suggests that for m=p+1, the theorem is valid. Now that we have established the BVPs' generality,

To achieve it, we set $m = 2$.

For n=1 and $k \geq 1, we\ have$

$$y^{(1+k)}(x)|_{x=0} = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!\,(-1)^{k_1}}{k_1!\,(k_2-k_1)!\,(k-k_2)!}y^{(k_2-k_1)}(x)y^{(k-k_2)}(x).$$

By definition, we have

$(n+k)!\,Y(n+k) = \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!(-1)^{k_1}}{k_1!(k_2-k_1)!(k-k_2)!}(k_2 - k_1)!\,Y(k_2-k_1)\,(k-k_2)(x).!\,Y(k-k_2),$

From (2.4), for n=1, we have

$Y(1+k) = \frac{k!}{(1+k)!}[\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\left(\frac{(-1)}{k!}\right)([\prod_{i=2}^{2}Y(k_{i=1} - k_{i=2}]Y(k-k_2)],$

$Y(1+k) = \frac{k!}{(1+k)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1}Y(k_2-k_1)Y(k - k_2).$

The theorem is true for n=1 because that is the needed outcome.

Now suppose the DT is as follows for n=q:

$Y(q+k) = \frac{k!}{(q+k)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1}Y(k_2-k_1)Y(k - k_2)$

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For $k \geq 1, we\ have$

$y^{(q+k)}(x)|_{x=0} =$

$\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!(-1)^{k_1}}{k!(k_2-k_1)!(k-k_2)}y^{(k_2-k_1)}(x)^{(k-k_2)}(x),$

$y^{(q+1+k)}(x)|_{x=0} =$

$\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!(-1)^{k_1}}{k!(k_2-k_1)!(k-k_2)}y^{(k_2-k_1)}(x)^{(k-k_2)}(x),$

$(q+1+k)!\,Y(q+1+k) =$

$\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{k!(-1)^{k_1}}{k!(k_2-k_1)!(k-k_2)}y^{(k_2-k_1)}(x)^{(k-k_2)}(x),$

From (2.4), for n =q+1, we have

$Y(q+1+k) =$

$\frac{k!}{(q+1+k)!}\left[\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\left(\frac{(-1)}{k_1}\right)([\prod_{i=2}^{2}Y(k_{i=1} - k_{i-2}]\,Y(k - k_2)\right)]$

$Y(q+1+k) = \frac{k!}{(q+1+k)!}\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{(-1)^{k_1}}{k_1}Y(k_2 - k_1)Y(k-k_2)$

Note that the theorem holds for n=q+1.

## Analysis of Higher-order Boundary Value Problems via Differential Transformation

Let's define the differential transform of the deflection function y(x) according to Eq. (2.1):

$$\mathrm{Y(k)} = \frac{1}{k!}\left[\frac{d^2y(x)}{dx^k}\right]_{x=x_0} \quad (3.1)$$

Where$x_0 = 0$, Additionally, the deflection function can be written as Y(k) from Eq (2.2)as:

$$\mathrm{y(x)} = \sum_{k=0}^{\infty}\left[\frac{x^k}{k!}\right]\mathrm{Y(k)}. \quad (3.2)$$

By, utilising the transformational operations created in Section 2, one can obtain the following recurrence equations by using the differential transforms of Equations (2.6)and (2.7) respectively. m=0,1,2…..

$$\mathrm{Y(2m+k)} = \sum_{k=0}^{\infty}\left[\frac{(2m)!}{(2m+k)!}\right]\mathrm{Y(.,.)}, \quad (3.3)$$

$$\mathrm{Y(2m+k+1)} = \sum_{k=0}^{\infty}\left[\frac{(2m+1)!}{(2m+k+1)!}\right]\mathrm{Y(.,.)}, \quad (3.4)$$

Where Y (.,.) denotes the function of f(x,yconverted), which may be linear or nonlinear. It should be observed that Eq is unaltered by the boundary conditions (3.2).

In the cases of even-order (odd-order) boundary value concerns, respectively, the differential transformations of the boundary conditions at x = 0 are determined from Eqs. (2.8) and (2.5), using the definition (3.1) as a guide:

the result is $j =$

$$Y(2j) = \frac{1}{(2j)!}\alpha_{2j},\ j = 0,1,2,\ldots\ldots,(2m-1), \quad (3.5)$$

$$Y(2j+1) = \frac{1}{(2j+1)!}\gamma_{2j+1}, \quad j = 0,1,2,\ldots\ldots,(2m), \quad (3.6)$$

When (3.3) or (3.4) in place of (3.5) and (3.6) and using (3.2)' gives for j=0,1,2,……(m-1)

$$\mathrm{y(x)} = \sum_{k=0}^{\infty}\left[\frac{1}{(2j)!}\alpha_{2j}\right],\ \mathrm{Y(k)x^k}, \quad (3.7)$$

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And for j=0,1,2,……..,m,

$$\mathrm{y(x)} = \sum_{k=0}^{\infty}\left[\frac{1}{(2j+1)!}\gamma_{2j+1}\right]\mathrm{Y(k)x^k} \quad (3.8)$$

Nothing that $y^{(2r+1)}(0) = A_r, r = 0,1,2,\ldots\ldots,(m-1), and\ y^{2r}(0) = B_r, r = 0,1,2,\ldots\ldots,m,$

Constants that will be roughly estimated at the final value of x=b

**Numeral Illustrations**

The differential transformation method will be used in this part to test both linear and nonlinear HOBVPs.

**Example 4.1.**

The following linear fifth-order BVP is the first to be studied, and the Adomian technique is also employed to resolve it.

$$y^{(u)}(x) = y(x) - 15e^{x}\_10xe^{x}, \qquad 0 < x1, \tag{4.1}$$

Based on the boundary circumstances

$$y(0) = 0, y'(0) = 1, y''(0) = 0, \; y(1) = -e \tag{4.2}$$

When Eq. (4.1) is subjected to the operations of (DT). The recurrence relation given below is attained:

$$\mathrm{Y}\,(\mathrm{k}\ +5) = \frac{\mathrm{k}![\mathrm{Y}(\mathrm{k}) - \frac{15}{\mathrm{k}!}10\left(\sum_{x=0}^{k}[\sum(\mathrm{k}-1-1)]\right)}{(\mathrm{km}+5)!} \tag{4.3}$$

When Eq. (4.1) is subjected to the operations of (DT), the following recurrence relation is attained.

$$Y(0) = 0,\ Y(1) = 1,\ Y(2) = 0 \tag{4.4}$$

From Eq. (2.1) $\alpha = \frac{y''(0)}{3!} = \mathrm{Y}(3)$and $\mathrm{b} = \frac{\mathrm{Y}^{iv}(0)}{4!} = \mathrm{Y}(4)$. Utlizing the converted B.C.'s in Eq. (5.3), the recursion relation in Eq (5.4), Y (k) for ≥ 5

The converted B.C.'s in Eq. (4.3), the recurrence relation in Eq (4.4) Y (k) for k≥ 5

Are simple to acquire.

By taking N=13 and evaluating the B.C.'s provided in Equ. Constants a and b. (4.2),

The system is established for x=1.

$$\frac{148284463}{148262400}a + \frac{3632669041}{3632428800}b = -\frac{4541061529}{5448643200}$$
$$\frac{2395958141}{79833600}a + \frac{5148284463}{37065600}b = -e + \frac{15028547}{129729600}$$

$this\ is\ turn\ gives\ a = -0.3333315065 and\ b = -0.5000018268.$

The following series' answer is obtained by applying the inverse transformation rule in Eq. is examined up to N=24 $Y(x) = x - 0.5000018296x^3 - 0.3333315040x^4 - 0.125x5 - 0.33333333335x^6 - 0.6944444444E - 2x^7 - 0.1190476464E - 2x^8 - 0.1736109901E - 3x^9 - 0.2201585538E - 4x^{10} - 0.2480158750E - 5x^{11} - 0.2505317839E - 6x^{12} - 0.2296443145E - 7x^{13} - 0.192508757E - 8x^{14} - 0.45911969220E - 9x^{15} - 0.1070602822E - 10x^{16} - 0.7169212599E - 12x^{17} - 0.2655261821E - 13x^{18} - 0.2655264821E - 11x^{19} - 0.1179714344E - 15x^{20} - 0.78.9303184E - 17x^{21} - 0.3914588213E - 18x^{22} - 0.1868326235E - 19x^{23} - 0.8509973664E - 21x^{24}$

**Example 4.2:** The following non-linear fifth order BVP is what we look at next.

$$\mathrm{y}^{\mathrm{v}}(\mathrm{x}) = \mathrm{e}^{-\mathrm{x}}\mathrm{y}^2(\mathrm{x}), \qquad 0$$

Depending on the boundary circumstances

$$y(0) = 0 = y'(0) = y''(0) = 1, y(1) = y'(1) = e$$

When Eq. (5.6) is subjected to the operations of (DT), the following recurrence is discovered:

$$Y(k+5) = \frac{k!}{(k+5)!}\sum_{l=0}^{k}\sum_{s=0}^{l}\frac{(-1)^s}{s!}\,Y(l-s)Y(k-l).$$

At x=0, the following converted BCs are possible produced by using Eqs. (2.1) and (4.7):

$$Y(0) = 1, Y(1), Y(2) = \frac{1}{2}$$

Where, according to Eq. (2.1), $a_1 = \frac{y''(0)}{3!}] = Y(3) and\ a_2 = \frac{y^{iv}(0)}{4!} = Y(4)$. Utilizing the converted B.C.'s in Eq. (4.4), Y (k) for x=1, and the recurrence relation in Eq. (4.3), with N = 12, to get the system:

$$e - \frac{80149541}{31933440} = \frac{98589}{98560}a_1 + \frac{1996097}{1996097}a_2 + \frac{1}{133056}a_1^2 + \frac{1}{47520}a_1a_2$$

$$e - \frac{5848303}{2851200} = \frac{285343}{95040}a_1 + \frac{665471}{166320}a_2 + \frac{1}{13860a_1^2} + \frac{1}{3960}a_1a_2$$

This in turn give $a_1$ =0.666611767 and $a_2$ =0.0416703271.

For N=20, these value are $a_1$ =0.1666611892 and $a_2$=0.0416703549.

We can assess the following series solution up to by using the inverse transformation rule in Eq (4.2). N =20:

$y(x) = 1 + x + 0.5x^2 + 0.1666611895x^3 + 0.04167033549x^4 + 0.83333333333E - 2x^5 + 0.1388888889E - 2x^6 + 0.19841269834E - 3x^7 + 0.2469995611E - 4x^8 + 0.265624214575E - 5x^9 + 0.2756771922E - 6x^{10} + 0.2505310920E - 7x^{11} + 02087772900E - 8x^{12} + 0.1605698298E - 9x^{13} + 0.114746900E - 10x^{14} + 0.3304439489E - 8x^{15} + 0.1987443795E - 8x^{16} + 0.2812694000E - 14x17 + 0.1560220001E - 15x^{18} + 0.8235580000E - 17x^{19} + 0.443090504E - 12x^{20}$

The numerical results of the differential transformation method for linear and nonlinear fifth-order BVPs are presented in Table 1 with comparisons to the exact solution.

| LN-5th Bvp | | | NLN5th Order BVP | |
|---|---|---|---|---|
| X | DTM(N=24) | Yexact=x(1-x)e^x | DTM(N=20) | Yexact= e^x |
| 0.0 | 0.000000001E+00 | 0.000000001E+00 | 0.100000000E+01 | 0.100000000E+01 |
| 0.1 | 0.994653900E-01 | 0.994653900E-01 | 0.110517100E+01 | 0.110517100E+01 |
| 0.2 | 0.195424400E+00 | 0.195424400E+00 | 0.122140300E+01 | 0.122140400E+01 |
| 0.3 | 0.358037900E+00 | 0.358037900E+00 | 0.134985900E+01 | 0.134985800E+01 |
| 0.4 | 0.412180200E+00 | 0.412180300E+00 | 0.149182500E+01 | 0.149182400E+01 |
| 0.5 | 0.412180200E+00 | 0.412180300E+00 | 0.164872100E+01 | 0.164872500E+01 |
| 0.6 | 0.437408400E+00 | 0.437408500E+00 | 0.182212100E+01 | 0.182212200E+01 |
| 0.7 | 0.422887900E+00 | 0.422887800E+00 | 0.201375200E+01 | 0.201375300E+01 |
| 0.8 | 0.356086300E+00 | 0.356086500E+00 | 0.223554000E+01 | 0.223554800E+01 |
| 0.9 | 0.221364000E+00 | 0.221365000E+00 | .2459604000E+01 | .2459604500E+01 |
| 1.0 | 0.302392000E-06 | 0.302394000E-06 | 0.271828000E+01 | 0.271827000E+01 |

**Example 4.3**The following linear sixth-order BVP is taken into consideration in the study, and it is also solved using the Adomian decomposition approach (ADM)

$y^{vi}(x) = y(x) = 6e^x, 0 < x < 1,$ (1)

Depending on the boundary circumstances

$y(0) = 1, y''(0) = -1, y^{iv}(0) = -3, y(1) = 0, y''(1) = -2e, y^{iv}(1) = -4e.$

$\mathrm{Y}(\mathrm{K}+6) = \frac{\mathrm{k}!\left[\mathrm{Y}(\mathrm{k})-\frac{6}{\mathrm{k}!}\right]}{[\mathrm{k}+6]!}$ (2)

At x=0, the altered BCs shown below can be obtained. Using Eqs (2.1) and (2 ):

$Y(0) = 1, Y(2) = -\frac{1}{2}, Y(4) = -\frac{1}{8},$

From Eq. (2.1), $\mathrm{a} = \mathrm{Y}(1), \mathrm{b} = \frac{y'''(0)}{3!} = \mathrm{Y}(3)$and $\mathrm{c} = \frac{\mathrm{y}^{\mathrm{v}}(0)}{5!} = \mathrm{Y}(5).$

Using the converted B.C.'s in Eq. (1) and the recurrence relation in Eq. (2), Y(k) for $k \geq 6$ are simple to acquire. By assuming N=13 and evaluating for x=1the B.C.'s in Eq., the constants a, b, and c, the system shown below is generated:

$$\frac{889750903}{889574400}a + \frac{60481}{60480}b + \frac{332641}{332640}c = \frac{456655181}{1245404160},$$
$$\frac{332641}{39916800}a + \frac{5041}{840}b + \frac{60481}{3024}c = -2e + \frac{110549143}{39916800},$$
$$\frac{60481}{362880}a + \frac{1}{20}b + \frac{5041}{43}c = -4e + \frac{829261}{121061},$$

This results in a = $-0.5489002288\mathrm{E}^{-6}$, b = $-0.3334428229$ and c = $-0.03334430492$.

N = 23 results in these numbers. a = $-0.4557205885\mathrm{E}^{-6}$b = $-0.3334428279$ and c = $-0.03334427191$.

The inverse transformation rule in Eq. (4.2) is then used to calculate the series solution up to N = 23 as follows:

$Y(x) = 1 - 0.4557205875E^{-6} - 6x - 0.5X^2 - 0.3333329279x^3 - 0.125x^4 - 0.03333327191x^5 - .6944444444E^{-2} - 2x^6 - .1190476283E^{-2} - 2x^7 - .1736111111E^{-3} - 3x^8 - .2204584867E^{-4} - 4x^9 - .2480158730E^{-5} - x^{10} - .2505208992E^{-6}x^{11} - .2296443269E^{-7}x^{12} - .1927085335E^{-8}x^{13} - .1491196928E^{-9}x^{14} - .1070602736E^{-10}x^{15} - .7169215999E^{-11}x^{16} - .4498329534E^{13}x^{17} - .2655265185E^{-14}x^{18} - .1479714382E^{15}x^{19} - .7809603484E^{-17}x^{20} - .3914587736E^{-18}x^{21} - .1868326192E^{-19}x^{22}.8509971524E^{-21}x^{23}$

**Example 4.4:** the 6th-order nonlinear BVP that follows is what we consider next.

$y^{iv}(x) = e^x y^2(x), \qquad 0 < x < 1,$ (4.1)

Depending on the boundary circumstances

$y(0 = 1, y'(0) = -1, y''(0) = 1, y(1) = e^{-1}, y'(1) = -e^{-1}y''(1) = e^{-1}$ (4.2)

When Eq (4.3) is subjected to DT procedures, the recurrence connection shown below is attained:

$\mathrm{Y}(\mathrm{k}+6) = \frac{\mathrm{k}!}{(\mathrm{k}+6)!}\sum_{l=0}^{k}\sum_{s=0}^{l}\frac{1}{s!}\mathrm{Y}(l-s)(\mathrm{k}-l).$ (4.3)

Using Eqs. (2.1) and (4.4), the following transformed B.C.'s at x = 0 can be derived:

$\mathrm{Y}(0) = 1, \mathrm{Y}(1) = -1, \mathrm{Y}(2) = \frac{1}{2}\mathrm{x}$ (4.4)

Where, according to Eq. (2.1), $a_1 = \frac{y'''(0)}{3!} = y(3), a_2 = \frac{y^{iv}(0)}{4!} = Y(4) and\ a_3 = \frac{y^{v}(0)}{5!} =$Y (5) . Using the converted B.C.'s in Eq. (4.4) and the recurrence relation in Eq. (4.5), Y (k) for k ≥ 6 are given .

$Y(6) = \frac{1}{720}, Y(7) = -\frac{1}{5040}, Y(8) = \frac{1}{40320}, Y(9) = \frac{1}{30240}a_1 + \frac{1}{362880},$

$Y(10) = \frac{1}{75600}a_2 - \frac{1}{3628800}, Y(11) = \frac{1}{166320}a_3 + \frac{1}{39916800},$

And so on.

By valuating the constants $a_1$, $a_2$, and $a_3$ using the B.C.'s in Eq. (4.3), X=1, and assuming N=11, the following system is produced:

$$e^{-1} - \frac{200701}{3991680} = \frac{30241}{30240}a_1 + \frac{75601}{75600}a_2 + \frac{166321}{166320}a_3,$$
$$-e^{-1} - \frac{321}{44800} = \frac{10081}{3360}a_1 + \frac{30241}{7560}a_2 + \frac{75601}{15120}a_3$$
$$e^{-1} - \frac{46943}{45630} = \frac{2521}{420}a_1 + \frac{10081}{840}a_2 + \frac{30241}{1512}a_3$$

This is turn give $a_1$=-1.666630261, $a_2$=.04166085689 and $a_3$=.008330914797.

For N =17, these values are $a_1$ =.1666633333, $a_2$=.03166152737 and $a_3$= -.009001283835. By utilising the inverse transformation rule in Eq., the following series solution is evaluated up to N=17

$y(x) = 1 + x + 0.5x\text{^}2 + 0.1666613333x\text{^}3 + 0.4416701527549E - 4x\text{^}4 - 0.83334283833E - 2x\text{^}5 + 0.1388888889E - 2x\text{^}6 - 0.19841269834E - 3x\text{^}7 + 0.246980125611E - 4x\text{^}8 - 0.275562421455E - 5x\text{^}9 + 0.2756551122E - 6x\text{^}10 - 0.2505378926E - 7x\text{^}11 + 02087772900E - 8x\text{^}12 - 0.16056904400E - 9x\text{^}13 + 0.4447469100E - 10x\text{^}14 - 0.7614535000E - 12x\text{^}15 - 0.3944923795E - 8x\text{^}16 - 0.7792669432E - 10x\hat{}\ 17$

**Table 2** Provides numerical results for the differential translation formation technique DTM for both linear and nonlinear sixth-order BVP problems with comparisons to the precise solution

| | LN 6th Order BVP | | NLN 6th Order BVP | |
|---|---|---|---|---|
| X | DTM(N=23) | $Y_{exact}$=(1-x)e^x | DTM(N=17) | $Y_{exact}$=e^x |
| 0.1 | 0.100000000E+01 | 0.100000000E+01 | 0.100000000E+01 | 0.100000000E+01 |
| 0.2 | 0.984553799E+00 | 0.984553799E+00 | 0.914937399E+00 | 0.914937399E+00 |
| 0.3 | 0.987222100E+00 | 0.987222100E+00 | 0.828833899E+00 | 0.828833899E+00 |
| 0.4 | 0.885084600E+00 | 0.885084600E+00 | 0.750818299E+00 | 0.750818299E+00 |
| 0.5 | 0.834367399E+00 | 0.834367399E+00 | 0.680330200E+00 | 0.680330200E+00 |
| 0.6 | .738947296E+00 | .738947296E+00 | .606530800E+00 | .606530800E+00 |
| 0.7 | 0.614123501E+00 | 0.614123501E+00 | 0.558911800E+00 | 0.558911800E+00 |
| 0.8 | 0.45500800E+00 | 0.45500800E+00 | 0.506688500E+00 | 0.506688500E+00 |
| 0.9 | 0.245860000E+00 | 0.245860000E+00 | 0.459310000E+00 | 0.459310000E+00 |
| 1.0 | -0.309816900E-04 | -0.309816900E-04 | 0.337779800E+00 | 0.337779800E+00 |

DTM (N=17) and N=23is DTM of order 17and 23 respectively.

## Graphical Explanation

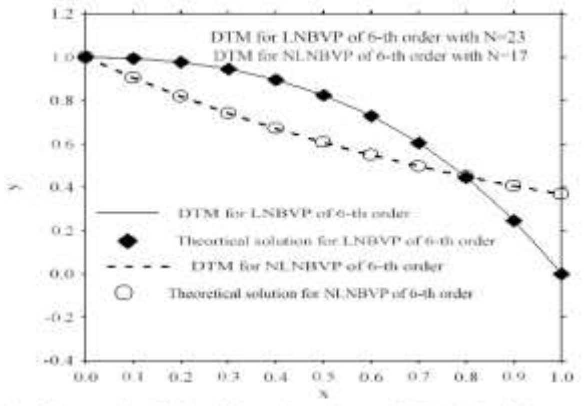


Fig. 2 Comparison differential transformation method (DTM) solution of linear and nonlinear BVP of 6-th order with exact solution

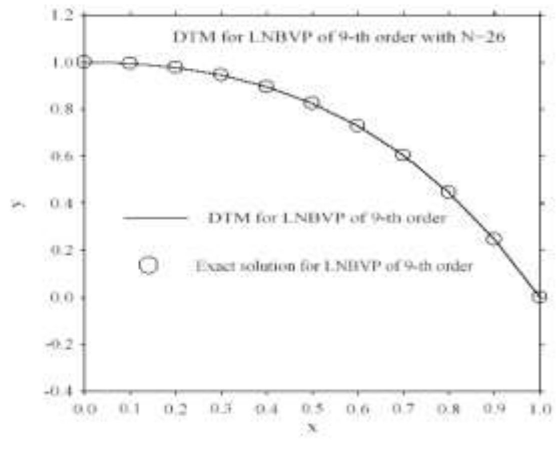


Fig. 3 Comparison differential transformation method (DTM) solution of linear BVP of 9-th order with exact solution y(x)=(1-x)exp(x)

## Conclusion

Maple 4 was used to carry out the calculations related to the aforementioned example. Agarwal's [1] guarantees the solution's existence and originality. First, we provided a definition of HOBVPs. Next, we provided a DTM analysis of HOBVPs. Finally, we provided a higher-order series solution for the aforementioned problems. It is established that the DTM is an extremely quick, accurate, and cost-effective approach for solving HOBVPs in limited domains.

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