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Research Article

Continuous Dunkl Wavelet Transform

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Abstract

Using the theory of the Dunkl transform and associated convolution, Dunkl wavelet transform (DWT) is defined. Certain boundedness results and inversion formula for the continuous DWT are obtained. Some properties related to the Dunkl wavelet transform are studied.

Keywords: *Dunkl transform, wavelet transform, Dunkl operator*

AMS Subject Classifications*: 42C40, 65T60, 44A35,65R10*

1. Introduction

The wavelet transform of a function $f \in L^2(R)$ with respect to the wavelet $\psi \in L^2(R)$ is defined by

$$\left(W_\psi f\right)(b,a) = \int_{-\infty}^{\infty} f(t)\overline{\psi_{b,a}(t)}\,dt,\ a,b \in R, a>0, \tag{1}$$

where

$$\psi_{b,a}(t) = a^{-1/2}\psi\left(\frac{t-b}{a}\right). \tag{2}$$

In terms of translation τ_b defined by

$\tau_b \psi(t) = \psi(t-b),\ b \in R$

and dilation D_a defined by

$$D_a\psi(t) = a^{-1/2}\psi\left(\frac{t}{a}\right),\quad a>0$$

.

we can write

$$\psi_{b,a}(t) = \tau_b\ D_a\ \psi(t). \tag{3}$$

From (1) and (3) it is clear that wavelet transform of the function f on R is an integral transform for which the kernel is the dilated translate of ψ.

We can also express (1) as the convolution:

$$\left(W_\psi f\right)(b,a) = \left(f * g_{o,a}\right)(b), \tag{4}$$

where

$$g(t) = \overline{\psi(-t)}.$$

Since associated with each integral transform there exists a special kind of convolution, one can construct wavelet transform corresponding to an integral transform using the associated convolution. Pathak and Dixit, have constructed Bessel wavelet transform by using the theory of Hankel transform. Wavelets on finite intervals involving solution of Sturm-Liouville system has been studied by U.Depczynski. Dunkl transform is another important transform for the construction of wavelet and wavelet transform.

Motivated from the above works we describe a new construction of wavelets by using the theory of Dunkl transform.

For $\alpha \geq -1/2$ and $\lambda \in C$, the initial value problem

$$\Lambda_\alpha(f)(x) = \lambda f(x),\quad f(0)=1,\ x \in R, \tag{5}$$

where

$$\Lambda_\alpha(f)(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

called Dunkl Operator has a unique solution $E_\alpha(\lambda x)$, called Dunkl kernel and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)}j_{\alpha+1}(i\lambda x),\quad x \in R, \tag{6}$$

where j_α is the normalized Bessel function of the first kind and order α defined by

$$j_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} =$$

$$\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n+\alpha+1)},\ z \in C \tag{7}$$

We can write for $x \in R$ and $\lambda \in C$

$$E_\alpha(-i\lambda x) =$$

$$\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_{-1}^{1}(1-t^2)^{\alpha-1/2}(1-t)e^{i\lambda xt}dt. \tag{8}$$

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on R, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1}|x|^{2\alpha+1}dx. \tag{9}$$

For every $1 \leq p \leq \infty$, we denote by $L^{p,\alpha} = L_p(d\mu_\alpha)$, the

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space of complex-valued functions *f*, measurable on *R* such that

$$\|f\|_{p,\alpha} = \left(\int_R |f(x)|^p \, d\mu_\alpha(x)\right)^{1/p} < \infty \quad if \quad p \in [1,\infty)$$

and $\|f\|_{L\infty,\alpha} = ess\sup_{x\in R} |f(x)|$ if $p = \infty$. (10)

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on *R*, which was introduced and studied in (Vagif S.Guliyev *et al*)).

The Dunkl transform F_α of a function $f \in L_{1,\alpha}(R)$, is given by

$$F_\alpha f(\lambda) = \hat{f}(\lambda) = \int_R E_\alpha(-i\lambda x) f(x) d\mu_\alpha(x) \quad ; \lambda \in R \tag{11}$$

An inversion formula for this transform is given by

$$F_\alpha^{-1}\left(\hat{f}(\lambda)\right) = \left(\hat{f}(\lambda)\right)^\vee = f(x) = \int_R E_\alpha(i\lambda x)\hat{f}(\lambda) d\mu_\alpha(\lambda) \tag{12}$$

An Parseval formula for this transform is given by

$$\int_{-\infty}^{\infty} f(x) g(x) dx = \int_{-\infty}^{\infty} \hat{f}(\lambda)\hat{g}(\lambda) d\mu_\alpha(\lambda) \tag{13}$$

2. Dunkl Translation and Convolution

In this section following (Vagif S.Guliyev *et al*) we define Dunkl translation and associated convolution and discuss their important properties.

To define Dunkl convolution $*_\alpha$ we need to introduce a special type of translation, called Dunkl translation. For this purpose we need the basic function

$$W_\alpha(x,y,z) = \left(1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\right)\Delta_\alpha(x,y,z) \tag{14}$$

$$\sigma_{x,y,z} = \begin{cases} \dfrac{x^2+y^2+z^2}{2xy}, & if \quad x, y \in R\setminus 0 \\ 0 & otherwise \end{cases}$$

Where.

And Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x,y,z) = \begin{cases} d_\alpha \dfrac{\left(\left[(|x|+|y|)^2 - z^2\right]\left[z^2 - (|x|-|y|)^2\right]\right)^{\alpha-1/2}}{|xyz|^{2\alpha}}, & if \quad |z| \in A_{xy}, \\ 0 & otherwise, \end{cases}$$

where $d_\alpha = \left(\Gamma(\alpha+1)\right)^2 / \left(2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})\right)$,

and $A_{x,y} = \left(\||x|-|y|\|, |x|+|y|\right)$.

Also $\int_R |W_\alpha(x,y,z)| d\mu_\alpha(z) \le 4.$ (15)

The Dunkl translation $\tau_x f(y)$ of $f \in L_{p,\alpha}(R)$, $1 \le p < \infty$ is defined as follows

$$\tau_x f(y) = f(x,y) = \int_R f(z) W_\alpha(x,y,z) d\mu_\alpha(z) \tag{16}$$

Lemma 1. For all $x \in R$ and $f \in L_{p,\alpha}(R)$, $p \ge 1$

(i) $\|\tau_x f\|_{p,\alpha} \le 4\|f\|_{p,\alpha}$. (17)

(ii) $(\tau_x f)\hat{}(\lambda) = E_\alpha(i\lambda x) f\hat{}(\lambda)$. (18)

(iii) Let $p, q, r \in [1,\infty[$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then Dunkl convolution of $f \in L_{p,\alpha}(R)$ and $g \in L_{q,\alpha}(R)$ is defined by

$$f *_\alpha g(x) = \int_R \tau_x f(-y) g(y) d\mu_\alpha(y) \tag{19}$$

Lemma 2. Let $p, q, r \in [1,\infty[$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $f \in L_{p,\alpha}(R)$ and $g \in L_{q,\alpha}(R)$. Then convolution $f *_\alpha g(x)$ satisfies the following norm inequality

(i) $\|f *_\alpha g\|_{r,\alpha} \le 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}$, (20)

Moreover for all $f \in L_{1,\alpha}(R)$ and $g \in L_{2,\alpha}(R)$, we have

(ii) $(f *_\alpha g)\hat{}(\lambda) = f\hat{}(\lambda) g\hat{}(\lambda)$ (21)

3. Dunkl Wavelet Transform

For a function $\psi \in L_{p,\alpha}(R)$, define the dilation D_a is given by

$$D_a\psi(x) = \psi(ax), \quad a \in R \tag{22}$$

Using the Dunkl translation and the above dilation, the Dunkl wavelet $\psi_{b,a}(x)$ is defined as follows

$$\psi_{b,a}(x) = \tau_b D_a \psi(x) = \tau_b \psi(ax) = \int_R \psi(az) W_\alpha(b,x,z) d\mu_\alpha(z), \quad b \in R \tag{23}$$

The integral is convergent by virtue of (18). Now, using the wavelet $\psi_{b,a}$ the Dunkl wavelet transform (DWT) of $f \in L_{q,\alpha}$, $\frac{1}{p} + \frac{1}{q} = 1$, is defined as follows:

$$(D_\psi f)(b,a) = \langle f(x), \psi_{b,a}(x)\rangle = \int_{-\infty}^{\infty} f(x)\overline{\psi_{b,a}(x)} d\mu_\alpha = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{\psi(az)} W_\alpha(b,x,z) d\mu_\alpha(z) d\mu_\alpha(x). \tag{24}$$

Provided the integral is convergent. Since by (16) and (17) $\psi_{b,a}(x) \in L_{p,\alpha}(R)$ whenever $\psi \in L_{p,\alpha}(R)$. By virtue of Lemma, the integral is convergent for $f \in L_{q,\alpha}$, $\frac{1}{p} + \frac{1}{q} = 1$.

The continuity and bundedness results follow from the following theorem.

Theorem 1. Let $f \in L_{p,\alpha}$ and $\psi(a.) \in L_{q,\alpha}, a > 0$ with $1 \le p,q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $(D_\psi f)(b,a)$ be the continuous Dunkl wavelet transform. Then

$$(i)\|(D_\psi f)(b,a)\|_{L_{r,\alpha}} \le 4\|f\|_{L_{p,\alpha}}\|\psi(a.)\|_{L_{q,\alpha}}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,\ 1 \le p,q,r < \infty \tag{25}$$

$$(ii)\|(D_\psi f)(b,a)\|_{\infty,\alpha} \le 4\|f\|_{L_{p,\alpha}}\|\psi(a.)\|_{L_{q,\alpha}}, \quad \frac{1}{p} + \frac{1}{q} = 1 \tag{26}$$

Proof (ii). We have

$$(D_\psi f)(b,a) =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{\psi(az)}\, W_\alpha(b,x,z)\, d\mu_\alpha(z)\, d\mu_\alpha(x)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{\psi(az)}\, W_\alpha^{1/p}(b,x,z) W_\alpha^{1/q}(b,x,z)\, d\mu_\alpha(z)\, d\mu_\alpha(x)$$

Using Holder's inequality, we get

$$\left|(D_\psi f)\ (\mathrm{b,a})\right| \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\mathrm{x})|^{\mathrm{p}} W_\alpha(b,x,z)\ \mathrm{d}\,\mu_{\ \alpha}(\mathrm{x})\ \mathrm{d}\,\mu_\alpha(\mathrm{z})\right)^{1/p}$$
$$\times \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(\mathrm{az})|^{\mathrm{q}} W_\alpha(b,x,z)\ \mathrm{d}\,\mu_{\ \alpha}(\mathrm{x})\ \mathrm{d}\,\mu_\alpha(\mathrm{z})\right)^{1/q}$$
$$\le \left(\int_{-\infty}^{\infty} |f(x)|^{p}\, d\,\mu_{\ \alpha}(\mathrm{x}) \int_{-\infty}^{\infty} W_\alpha(b,x,z)\ \mathrm{d}\,\mu_\alpha(\mathrm{z})\right)^{1/p}$$
$$\times \left(\int_{-\infty}^{\infty} |\psi(az)|^{q}\, d\,\mu_{\ \alpha}(\mathrm{x}) \int_{-\infty}^{\infty} W_\alpha(b,x,z)\ \mathrm{d}\,\mu_\alpha(\mathrm{x})\right)^{1/q}$$

$$\le 4^{\left(1/p+1/q\right)}\left(\int_{-\infty}^{\infty}|f(x)|^{p}\ \mathrm{d}\mu_\alpha(\mathrm{x})\right)^{1/p} \times \left(\int_{-\infty}^{\infty}|\psi(\mathrm{az})|^{\mathrm{q}}\ \mathrm{d}\,\mu_\alpha(\mathrm{z})\right)^{1/q}$$
$$\le 4\ \|f\|_{\mathrm{L}_{p,\alpha}}\ \|\psi(a.)\|_{\mathrm{L}_{q,\alpha}}.$$

The inequality (i) follows from 21.

Theorem 2. (Parseval's Formula for Dunkl wavelet transform).

If $\psi \in L_{2,\alpha}$ so that $\mathrm{C}_\psi = \int_{-\infty}^{\infty}\frac{|\hat{\psi}(\lambda)|^2}{|\lambda|}d\mu_\alpha(\lambda) < \infty$,

Then for any $\mathrm{f,g} \in L_{2,\alpha}$, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(D_\psi f)(b,a)\ \overline{(D_\psi g)(b,a)}\ \frac{d\mu_\alpha(a)\,d\mu_\alpha(b)}{a} = C_\psi \int_{-\infty}^{\infty} f(t)g(t)d\mu_\alpha(t) \tag{27}$$

Proof. By Parseval's Formula for Dunkl transform, we have

$$(D_\psi f)(\mathrm{b,a}) = \int_{-\infty}^{\infty} f(x)\overline{\psi}_{b,a}(x)\, d\mu_\alpha(x)$$
$$= \langle \mathrm{f(x)},\ \psi_{\mathrm{b,a}}(x)\rangle$$
$$= \left\langle \hat{f}(\lambda), \hat{\psi}_{b,a}(\lambda)\right\rangle$$
$$= \int_{-\infty}^{\infty}\hat{f}(\lambda)\hat{\psi}_{b,a}(\lambda)\,\mathrm{d}\,\mu_\alpha(\lambda)$$
$$= \int_{-\infty}^{\infty}\hat{f}(\lambda)\hat{\psi}(a\lambda)\,E_\alpha(i\lambda b)\,\mathrm{d}\,\mu_\alpha(\lambda)$$
$$= \left(\hat{f}(\lambda)\hat{\psi}(a\lambda)\right)^{\vee}(b).$$

Similarly $\overline{(D_\psi g)(b,a)} = \overline{\left(\hat{g}(\lambda)\hat{\psi}(a\lambda)\right)^{\vee}(b)}$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(D_\psi f)(b,a)\ \overline{(D_\psi g)(b,a)}\ \frac{d\mu_\alpha(a)\,d\mu_\alpha(b)}{a} =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\hat{f}(\lambda)\hat{\psi}(a\lambda)\right)^{\vee}(b)\ \overline{\left(\hat{g}(\lambda)\hat{\psi}(a\lambda)\right)^{\vee}(b)}\ \frac{d\mu_\alpha(a)\,d\mu_\alpha(b)}{a}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\hat{f}(\lambda)\hat{\psi}(a\lambda)\right)\ \overline{\left(\hat{g}(\lambda)\hat{\psi}(a\lambda)\right)}\ \frac{d\mu_\alpha(a)\,d\mu_\alpha(\lambda)}{a}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(\lambda)\,\hat{g}(\lambda)\left|\hat{\psi}(a\lambda)\right|^2\ \frac{d\mu_\alpha(a)\,d\mu_\alpha(\lambda)}{a}$$
$$= \int_{-\infty}^{\infty}\left|\hat{\psi}(a\lambda)\right|^2\ \frac{d\mu_\alpha(a)}{a}\int_{-\infty}^{\infty}\hat{f}(\lambda)\,\hat{g}(\lambda)\,d\mu_\alpha(\lambda)$$
$$= \int_{-\infty}^{\infty}\frac{\left|\hat{\psi}(\lambda)\right|^2}{|\lambda|}d\mu_\alpha(\lambda)\int_{-\infty}^{\infty}\hat{f}(\lambda)\,\hat{g}(\lambda)\,d\mu_\alpha(\lambda)$$
$$= C_\psi\int_{-\infty}^{\infty} f(t)g(t)d\mu_\alpha(t).$$ This completes the proof.

Theorem 3. (Inversion Formula). If $f \in L_{2,\alpha}$ then

$$f(x) = \frac{1}{C_\psi}\int\int(D_\psi f)(b,a)\ \psi_{b,a}(x)\ \frac{d\mu_\alpha(a)\,d\mu_\alpha(b)}{a} \tag{28}$$

where $\mathrm{C}_\psi = \int_{-\infty}^{\infty}\frac{|\hat{\psi}(\lambda)|^2}{|\lambda|}d\mu_\alpha(\lambda)$.

Proof. For any $g \in L_{2,\alpha}$, we have from (28)

$$C_\psi\int_{-\infty}^{\infty} f(x)g(x)d\mu_\alpha(x)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(D_\psi f)(b,a)\ \overline{(D_\psi g)(b,a)}\,\frac{d\mu_\alpha(a)\,d\mu_\alpha(b)}{a}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(D_\psi f)(b,a)\int_{-\infty}^{\infty}\mathrm{g(x)}\ \overline{\psi}_{\mathrm{b,a}}(x)\ \mathrm{d}\,\mu_\alpha\ (\mathrm{x})\ \frac{d\mu_\alpha(a)\,d\mu_\alpha(b)}{a}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(D_\psi f)(b,a)\psi_{b,a}(x)\frac{d\mu_\alpha(a)\,d\mu_\alpha(b)}{a}\,g(x)\,d\mu_\alpha(x)$$

Therefore

$$f(x) = \frac{1}{C_\psi}\int\int(D_\psi f)(b,a)\psi_{b,a}(x)\frac{d\mu_\alpha(a)\,d\mu_\alpha(b)}{a}$$

This completes the proof.

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