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Research Article

Analysis of Buckling of Piles Fully Embedded in Ground According to Finite Element Method

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Abstract

This work intends to present the analysis of buckling of piles fully embedded on ground and subject to axial force in compression based on the Finite Element Method. The approach of eigen values is used for calculation of buckling forces of piles within this method. To calculate the critical buckling force by use of Finite Element Method the SAP 2000 software is utilized where a reinforced concrete pile of circular cross section, D=30cm, concrete class C-25/30, of 5m length, the modulus of elasticity of soil, k=1000kN/m², with pinned end conditions (p-p) on the head and on the tip, is considered. The constant modulus of soil reaction is adopted throughout the depth. The comparison of amount of the pile buckling force obtained from software SAP 2000 with its theoretical value and by Euler's force is done.

Keywords: *Finite Element Method, Pile, Pile buckling, Pile buckling force, Pile buckling length, Stiffness matrix.*

1. Introduction

The fundamental principle this method is based on, is the division of the body or of the space considered (under consideration) on finite elements in such a way as, each element being analyzed as an izolated member as well as by assuming a relationship inbetween (interconnection between), is then analyzed as an integral body. This method of analysis starting from particular to general or, from individual to universal, where a general conclusion is drawn by analysing particular elements is known as inductive approach. This approach is widely used in resolving engineering problems when a closed solution is hardly obtained.

The Finite Element Method is the most effective way for resolving (analyzing) beams supported on an elastic base. It is more effective (convenient) than the method of finite diferences, since the last mentioned requires quite diferent formulation of the equations for the ends and the end conditions, whilst division of the analyzed body in elements of various lengths is particularly difficult when this method is used.

Within the Method of Finite Elements, the continuum of endless (infinite) degrees of freedom is substituted with a discrete system of finite degrees of freedom and is then analyzed by a discrete analysis method. With regard to (concerning) mathematical formulation, the analyzed problem passed from the field of analysis to the field of algebrae. This method is used with the aim of valuing the buckling force of the pile fully embeded onthe ground and subject to only compresion force by using the beam element formulation.

2.Analysis of of pile buckling and determination of critical force

2.1 Determination of pile buckling force when k=const

The problem of pile buckling is closely related to the problem of a beam on an elastic base (L., Hetenyi, M.I., 1960). If an elastic flexural beam is subject to axial force then this force will have an effect on the beam deformation too, therefore with an aim to calculate a response of a vertical pile fully embedded on ground and externally loaded with an axial force, the pile will be considered (treated) as a beam on an elsatic foundation (Matlock, H. and L. C. Reese).

The flexure of the beam/column due to the vertical-axial force is closely related to the buckling force of a pile (Timoshenko, S. P. and J. M. Gere, 1961). The basic equation of the pile flexure subject to axial force is as follows:

$$E \cdot I \frac{d^4 y}{dx^4} + P \frac{d^2 y}{dx^2} + k \cdot y = 0 \qquad (1)$$

where:
E– modulus of elasticity of the pile (kN/m²)
I– second moment of inertia of the pile (m⁴)
P– axial force on pile (kN)
k- modulus of soil reaction (kN/m²), and
y- pile displacement (m)

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For the pile pinned supported on both ends, the end conditions will be:

$$y(x=0)=0;\quad y^{''}(x=0)=0$$
$$y(x=l)=0;\quad y^{''}(x=l)=0 \qquad (2)$$

The solution of the differential equation is:

$$y(x)=C\cdot\sin\left(\frac{n\cdot\pi\cdot x}{l}\right) \qquad (3)$$

The solution of differential equation for the force *(P)* is:

$$P_{krit}=n^2\frac{\pi^2}{l^2}\cdot E\cdot I+\frac{1}{n^2}\cdot\frac{k\cdot l^2}{\pi^2} \qquad (4)$$

where is:

P_E -Euler's force for column made of elastic material:

$$P_E=\frac{\pi^2}{l^2}\cdot E\cdot I \qquad (5)$$

The first part of the equation (4) corresponds to Euler's equation for buckling of column while its second part reflects the contribution of lateral restraint caused by the surrounding soils. In equation (4) the relation force – deflection is taken as linear and the modulus of soil horizontal reaction is constant with the depth, and for most of soft soils this is not a case.

To determine the minimum amount for P_{krit}, the *n* values (Figure 1) should be investigated for each value of *l*.

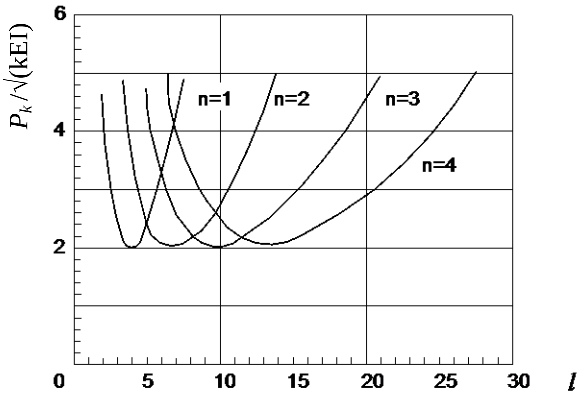

Figure 1 Buckling force P_k, in a non-dimensional form and in function of pile length *l* and *n*

For the pile embedded in an elastic medium, the general expression for critical buckling force is:

$$P_{krit}=P_E\left(n^2+\frac{B}{n^2}\right) \qquad (6)$$

where *n* is the number of sinusoidal semi-wave in a curved form of the column,

$$B=\frac{k\cdot l^4}{\pi^4\cdot E\cdot I} \qquad (7)$$

It is easy to show that minimum value of critical force P_{krit} corresponds to the below value:

$$n=\left[k\cdot(E\cdot I)^3\right]^{1/4} l/\pi\cdot E\cdot I=B^{1/4} \qquad (8)$$

For the pile with end conditions pinned-fixed, the Euler's force is:

$$P_E=\frac{\pi^2}{4\cdot l^2}\cdot E\cdot I \qquad (9)$$

By substituting *n* to P_E, in equation (6), after certain adjustment the following is obtained:

$$P_{krit}=\frac{1}{2}(k\cdot E\cdot I)^{1/2} \qquad (10)$$

The critical buckling length will be determined out of the condition:

$$\frac{\partial P}{\partial\left(\frac{l}{n}\right)}=0\Rightarrow l_0=\pi\cdot\sqrt[4]{\frac{E\cdot I}{k}}\;[m] \qquad (11)$$

Out of hereby it is seen that the buckling length or as it is called, the critical length, contains the following two elements: the pile element expressed through pile stiffness, *EI*, and the soil element expressed through design coefficient of soil reaction, *k*. If the pile is to stiff comparing to the soil, then the buckling length reaches relatively high values meaning that the pile load will cause considerable deflection. In contrary, if the soil is too stiff against the pile, then the buckling length of the pile would be relatively small. In this way it is possible that for the system pile – soil the buckling length to be determined and out of the solution of the given problem the point in which the deflection of pile is equal to zero and underneath of which the deflection amount is so small such as the same may be neglected. If the pile is shorter that it's buckling length that it is called a short pile, whilst the pile with its length longer than its buckling length is called a slender column. With other words, as higher the ratio between pile stiffness and soil (stiffness), the bigger pile length is required in order to consider it flexible.

For calculation of critical buckling force of the pile pinned at both ends, it is possible that in certain cases the equation (Davisson, M. T., July 1963) is used:

$$P_{krit}\geq 2\cdot\sqrt{k\cdot E\cdot I} \qquad (12)$$

Partcularly for higher values of *l*, the critical buckling force is calculated relatively high accuracy according to expression (Shnell, W., Czerwenka, G.):

$$P_{krit}\approx 2\cdot\sqrt{k\cdot E\cdot I} \qquad (13)$$

2.2 Determination of pile buckling force according to Finite Element Method

The solution of diferencial equation (1) according to finite element method in a more convenient form may be obtained if *Galerkins Method* (Heubner, K. H., and Thornton, E. A., 1981), is used, so the equation (1) will then be converted in bellow shown shape:

$$\sum_j \left\{ \int EIN_i''N_j''dx + \int PN_i'N_j'dx + \int kN_iN_jdx \right\} y_j = 0 \tag{14}$$

where is:

y_j – six nodal dispacements of the beam $u_1, v_1, \theta_1, u_2, v_2, \theta_2$ given in Fig. 2.

Based on equation (14) the pile buckling length is easyly resolved by use of Finite Element Method for end conditions, self weight, material nonlinearity and for the soil nonlinear effect (Cook,R. D., Malkus, D. S. And Plesha, M. E., 1989).

As shown in Fig. 2, the pile is separated in finite elements the vectors of which present axial displacement and bendings letered {u} dhe{v}, respectively:

$$\{u\} = \{u_1 \quad u_2\}^T, \quad \{v\} = \{v_1 \quad \theta_1 \quad v_2 \quad \theta_2\}^T \tag{15}$$

These deflection components may be assembled in a column vector {y}:

$$\{y\} = \{u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad \theta_2\}^T \tag{16}$$

N_j are six function shapes resulting out of these deflections, $u_1, v_1, \theta_1, u_2, v_2, \theta_2$.

Lets denote with $U_0(x)$ dhe $V_0(x)$ the axial and flexural displacements of any pint along x axis, respecttively:

$$\begin{aligned} U_0(x) &= N_a\{u\} \\ V_0(x) &= N_b\{v\} \end{aligned} \tag{17}$$

Where, N_a is a functional shape that defines the linear interpolation of $U_0(x)$ inbetween nodes, and N_b, includes (encompasses) the cubic function of the beam of the linear interpolation (Rao, S. S., 1989). These shapes of functions are given with the bellow shown expressions and are also shown in Fig.2 where is:

$$N_1 = 1 - \frac{x}{L}, \qquad N_4 = \frac{x}{L}$$

$$N_2 = 1 - \frac{3x^2}{L^2} + \frac{3x^3}{L^3}, \qquad N_3 = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$

$$N_5 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}, \qquad N_6 = \frac{x^3}{L^2} - \frac{x^2}{L}$$

The first integral in the above sum represents the elements of stiffness matrix, K_E, the second integral represents elements of stiffness geomemtric matrix, K_G, and the third integral represents elements of soil stiffness matrix, K_S. Therefore the matrix ij element is calculated as follows:

$$K_{E,ij} = \int EI \cdot N_i'' \cdot N_j''dx \tag{18}$$

$$K_{G,ij} = \int P \cdot N_i' \cdot N_j'dx \tag{19}$$

$$K_{s,ij} = \int k \cdot N_i \cdot N_j dx \tag{20}$$

By resolving the above equations (18), (19) and (20), the linear stiffness matrix K_E, the geometric stiffness matrix K_G, and the soil stiffness matrix K_S, take the following shapes:

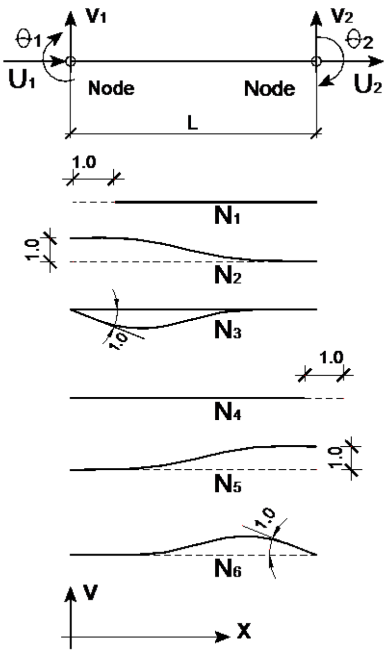

Figure 2 Node displacements and function shapes

$$[K_E] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{21}$$

$$[K_G] = -P^* \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5L} & \frac{1}{10} & 0 & -\frac{6}{5L} & \frac{1}{10} \\ 0 & \frac{1}{10} & \frac{2L}{15} & 0 & -\frac{1}{10} & -\frac{L}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5L} & -\frac{1}{10} & 0 & \frac{6}{5L} & -\frac{1}{10} \\ 0 & \frac{1}{10} & -\frac{L}{30} & 0 & -\frac{1}{10} & \frac{2L}{15} \end{bmatrix} \tag{22}$$

$$[K_s]=-\frac{L}{420}\begin{bmatrix}0&0&0&0&0&0\\0&156k&22kL&0&54k&-13kL\\0&22kL&4kL^2&0&13kL&-3kL^2\\0&0&0&0&0&0\\0&54k&13kL&0&156k&-22kL\\0&-13kL&-3kL^2&0&-22kL&4kL^2\end{bmatrix} \quad (23)$$

The total stiffness of a beam element of an elastic continual foundation is the sum of the matrixes $[K_E]$ (equation 21), $[K_G]$ (equation 22) dhe $[K_s]$ (equation 23):

$$[K]=[|K_E+K_s|+K_G]$$

$$[K]=\begin{bmatrix}k_{11}&k_{12}&k_{13}&k_{14}&k_{15}&k_{16}\\k_{21}&k_{22}&k_{23}&k_{24}&k_{25}&k_{26}\\k_{31}&k_{32}&k_{33}&k_{34}&k_{35}&k_{36}\\k_{41}&k_{42}&k_{43}&k_{44}&k_{45}&k_{46}\\k_{51}&k_{52}&k_{53}&k_{54}&k_{55}&k_{56}\\k_{61}&k_{62}&k_{63}&k_{64}&k_{65}&k_{66}\end{bmatrix} \quad (24)$$

Where is:

$$k_{11}=k_{44}=-k_{14}=-k_{41}=\frac{AE}{L}$$

$$k_{22}=k_{55}=\frac{12EI}{L^3}-\frac{6P^*}{5L}+\frac{156kL}{420}$$

$$k_{23}=k_{32}=-k_{56}=-k_{65}=\frac{6EI}{L^2}-\frac{P^*}{10}+\frac{22kL^2}{420}$$

$$k_{25}=k_{52}=-\frac{12EI}{L^3}+\frac{6P^*}{5L}+\frac{54kL}{420}$$

$$k_{26}=k_{62}=\frac{6EI}{L^2}-\frac{P^*}{10}-\frac{13kL^2}{420}$$

$$k_{33}=\frac{4EI}{L}-\frac{2P^*L}{15}+\frac{4kL^3}{420}$$

$$k_{35}=k_{53}=-\frac{6EI}{L^2}+\frac{P^*}{10}+\frac{13kL^2}{420}$$

$$k_{36}=k_{63}=\frac{2EI}{L}+\frac{P^*L}{30}-\frac{3kL^3}{420}$$

$$k_{66}=\frac{4EI}{L}-\frac{2P^*L}{15}-\frac{4kL^3}{420}$$

$$k_{12}=k_{13}=k_{15}=k_{16}=k_{42}=k_{43}=k_{45}=k_{46}=k_{51}=$$
$$k_{54}=k_{61}=k_{64}=k_{21}=k_{24}=k_{31}=k_{34}=0$$

In all above shown matrixes, *AE* represents the axial stiffness of the element, P the applied force on the element, and k, stiffness of springs that substitute the soil and that are equivalent with the lateral rezistance coeficient of soil. Is elastic springs are assumed at both sides of the pile, then the components of $[K_s]$ matrix in equation (24) shall be multiplied with $[2K_s]$.

The common way of determining critical buckling force of a beam-column element according Finite Element Method is determination of square roots of polinom determined (resulting) from the matrix [K] determinant.

The SAP software is used to determine the critical buckling force according to FEM.

A reinforced concrete pile of circular cross section is given and with the following end conditions: pinned in the bottom and in the head (*p-p*) (Figure 3). Given data:

D=30cm,
L=5m,
C25/30,
$K=100t/m^2=100kN/m^2$,
$I=3.974*10^{-4}m^4$
For *C25/30* we read: $E_{cm}=30.5kN/mm^2$
The pile is divided in "n" equal pieces length of Δx. The modulus of soil reaction for each node is calculated as in Fig. 3.
Then the modulus of soil reaction in certain points of the pile for $k=100\ t/m^2=1000\ kN/m^2$ will be:

$$K_1=\frac{1}{2}k_0\cdot\Delta x=\frac{1}{2}100\cdot 0.5=25t/m'$$

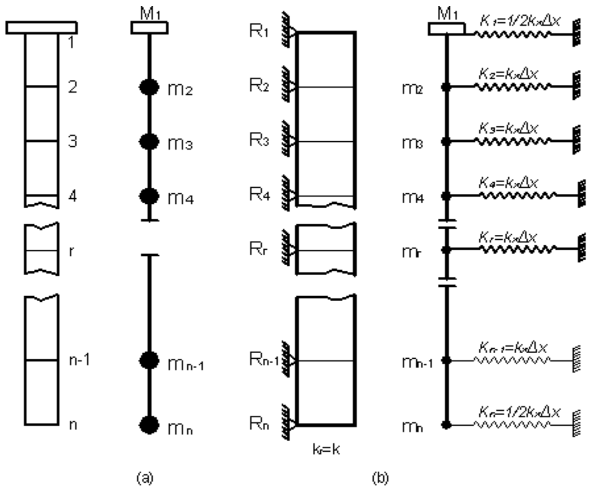

Figure 3 Discretization of soil-pile interaction diagram. (a) Idealization of pile structure, (b) soil modulus considered constant within the depth (Chandrasekaran, 1974).

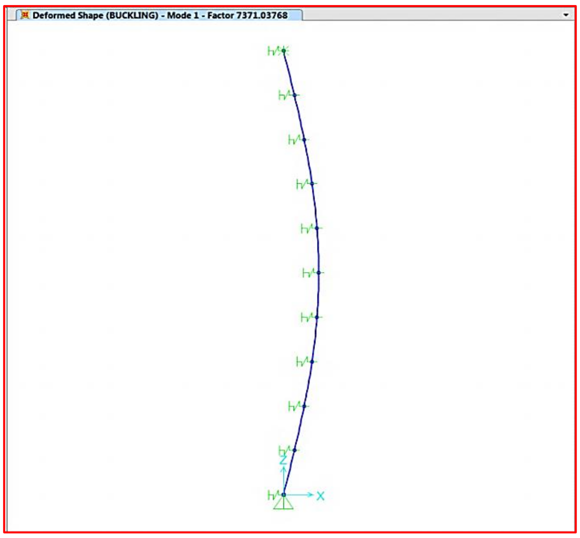

Figure 4 First buckling shape of the pile according to SAP 2000

$$K_2 = K_2 = K_3 = K_4 = K_5 =$$
$$K_6 = K_7 = K_8 = K_9 = k_0 \cdot \Delta x = 100 \cdot 0.5 = 50t/m^{'}$$
$$K_{10} = \frac{1}{2}k_0 \cdot \Delta x = \frac{1}{2}100 \cdot 0.5 = 25t/m^{'}$$

By use of SAP 2000 software for the pile, length of 5m, and of *=100 t/m²=1000 kN/*m^2, the critical buckling force obtained is P_{krit} = 7371.03 kN –according to SAP 2000 (Fig. 4). The pile buckling length is:

$$l_0 = 3.14\sqrt[4]{\frac{1212089}{1000}} = 5.858m$$

Due to the buckling length being bigger than the pile length, it is said that we have to do with the stiff pile.
Comparision of Eulers force for a pile pinned supported at both ends *(k=0)* with a pile buckling force of a pile of support conditions same as those of the column:

$$P_E = 4780{,}28kN < P_{krit} = 7371{,}03kN$$

As it is seen, the pile buckling force is for *54.20%* greater than the Eulers force.
The theorical value of the buckling force for the case when n=1, is:

$$P_{krit} = n^2\frac{\pi^2}{l^2}\cdot E\cdot I + \frac{1}{n^2}\cdot\frac{k\cdot l^2}{\pi^2} = 7315{,}885kN$$

Conclusions

In this paperwork, the way of determination of buckling force according to the Finite Element Method is presented. The model is based on formulation of pile stiffness matrix that is considered as a beam of an elastic foundation, while the soil is considered in one side of the pile only by being substituted with a series of elastic springs of the same stiffness as the stiffness of soil. The buckling force is determined is done by use of software SAP 2000. If a buckling force obtained through SAP 2000 with its theoretical value the conclusion is that these two values are very close to each other (the pile buckling force obtained by SAP 2000 is for *0.754%* higher than its theoretical value). Due to the buckling length being bigger than the pile length, we can say that we are dealing with a stiff pile.

The pile buckling force is for *54.20%* bigger than the Euler's force due to lateral limitations caused by surrounding soils.

The Finite Element Method may be used for calculation of buckling force of piles of different cross sections, different support conditions, and of different soil stiffness.

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