Research Article

Application of B–Spline Based FEM to One-Dimensional Problems

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Abstract

In this work, an attempt is made to use the B-spline basis functions as the shape functions in the finite element method. The linear B-Spline basis functions using two control points and an open uniform knot vector at a time are identical with the FEM shape functions. Hence, the open uniform linear basis functions can be used as a shape functions in the FEM. These basis functions are employed in the Galerkin's approximation For the spatial discretization. Several test cases are considered to study the effectiveness of the present method. The results obtained by the present method are compared and found to be in good agreement with the analytical solution and the finite element method.

Keywords: B-Spline FEM, Isogeometric Method, Eigenvalue problem, Galerkin Method, Open Uniform Knot vector.

1. Introduction

In the recent years, a new class of approximating methods which are variants of the finite element methods are proposed to solve various initial and boundary value problems. The methods that can be included in this category are Meshfree methods (Liu.G.R, 2009); B-Spline based Finite element method and Isogeometric methods (Hughes, et al, 2005, 2006). In these methods, the approximating function provides higher order of continuity and is capable of providing accurate solutions with continuous gradients throughout the domain.

In the present work, an attempt is made to use an approximating function for the field variable based on the B-Spline basis function to solve the various boundary value problems. A B-spline basis function is a piecewise polynomial function defined in terms of a parameter, the degree of which is independent of the number of control points. The parameter variable and the control points are related by the knot vector. An open uniform knot vector is used to obtain the first degree B-Spline basis function. For the spatial discretization, the Galerkin's approximation method (Reddy, 2005) is employed. Numerical studies are performed with problems of linear elasticity and Heat Conduction in one dimension. The problems considered are the extension of a prismatic bar due to body load, 1D Laplace eigenvalue problem and temperature distribution in a rectangular fin.

2. B–Spline Finite Element Method

The B-splines are a standard tool for describing and

modelling curves and surfaces in computer aided design and computer graphics. The aim of this section is to present a short description of B-splines and its associated terminology.

2.1 B-Spline Basis Function

The **cox-de Boor** recursion formula for the B-Spline basis functions (Rogers and Adams, 2002) are defined recursively starting with the zeroth degree ($p = 0$). These basis functions are given as

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

And for any polynomial degree ($p \geq 1$),

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad (1)$$

The ratio of the form 0/0 is defined as zero when evaluating these B-Spline basis functions. In the above equations, the basis functions are defined over a parametric domain ξ. The span of the parametric domain is known as the knot vector $\{\xi_1 \quad \xi_2 \quad . \quad . \quad . \quad \xi_{n+p+1}\}$, where ξ_i is the 'i' th knot, n is the number of basis functions and p is the polynomial degree.

A knot vector is a sequence in ascending order of parameter values. The shape of the basis functions is dependent on the knot spacing rather than the actual knot value. A knot vector is said to be open if its first and last knots have multiplicity equal to the polynomial order plus one. An important property of open knot vectors is that the resulting basis functions are interpolatory at the ends of

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the parametric space. If the knots are evenly spaced then knot vector is called uniform otherwise it is non-uniform.

As examples, for polynomial degree p = 0 and 1 using uniform knot vector {0 1 2 3 4 5 ...}, the basis functions are shown in figure 1 and figure 2.
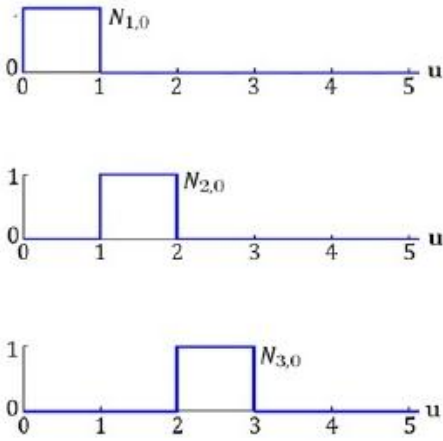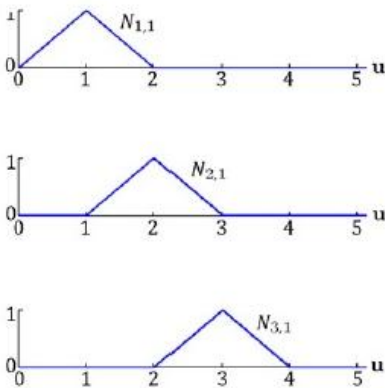

Fig. 1: B-Spline Basis for Zero

Fig. 2: B-Spline Basis for First

It can be observed from above figure 2 that linear basis functions using two control points at a time are identical with the FEM shape functions. Hence, the open uniform linear basis functions can be used as a shape functions in the FEM formulations.

3. The Test Problems

Three test problems are considered to study the effectiveness of the present method. They are a homogeneous bar subjected to distributed load acting along the longitudinal axis, one dimensional eigenvalue problem of finding the frequencies of longitudinal vibration of a rod and temperature distribution in a rectangular fin.

3.1 A Homogeneous Bar with Distributed Load

A homogeneous bar of length L, which is subjected to a distributed body load 'f = x', is considered as a first test case. The mathematical model for the physical system

shown in figure 3 is represented by one dimensional Poisson equation that is given by,

$$\frac{d^2u}{dx^2} + x = 0 \ , \ 0 \le x \le 1 \tag{2}$$

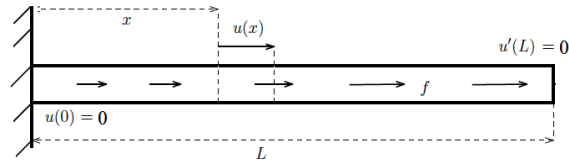With boundary conditions, $u(0) = 0 \ \& \ u'(1) = 0$,

Fig.3 Homogeneous Bar with Distributed Load

The longitudinal displacement u(x) that is produced by the force is the solution. In the B-spline based Finite element method (BSFEM), the displacement field is approximated by,

$$u(x) = \sum_{i=1}^{ne} N_{i,p}(x)u_i \tag{3}$$

In equation (3), 'ne' represents the number of nodes in an elements, $N_{i,p}$ is the basis functions obtained from the B-Spline basis functions for any point in the domain. The nodal displacements are represented by u_i. In the present study, only the first degree (p = 1) approximation is considered. Generally, the B-spline basis functions are not interpolatory except when the knot vector is open uniform and the basis functions are linear. When an open uniform knot vector is used, the essential boundary conditions can be directly implemented by using elimination approach. Therefore, an open uniform knot vector is taken as parametric space coinciding with the coordinates of the domain.

The weak form for the governing equation (2) is obtained by using weigh function W,

$$\int_0^1 W\left(\frac{d^2u}{dx^2} + x\right)dx = 0 \tag{4}$$

Integrating by parts and using the approximating function (eqn.2) and the boundary conditions, a linear system of equations are obtained as **KU** = **F** where,

$$K = \left[\int_0^1 \frac{dN^T}{dx}\frac{dN}{dx}dx\right]; \ U = \{u\}; \ F = \left[\int_0^1 xN^T dx\right] \tag{5}$$

Where N represents the B-spline basis functions. The stiffness matrix (K) and the force vector (F) are evaluated by using two point gauss quadrature. The domain is discretised with 6 nodes and 5 elements with two nodes per each element. The knot vector is developed using these nodes as control points for the knot coordinated system then each knot vector range is mapped to a parametric coordinated system. The obtained results are compared with the exact solution given by

$$u_{ex}(x) = \frac{x}{2} - \frac{x^3}{6} \tag{6}$$

The figure 4 shows the displacement field along the length of the bar. It can be observed from the figure that the results obtained by the present method are in very good agreement with the analytical solutions.
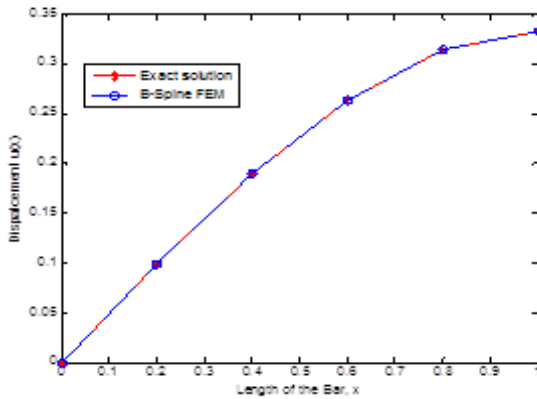

Fig.4. Comparison of field variable with exact solution

3.2 A one-dimensional eigenvalue problem

In this section, BSFEM is applied to a 1D Laplace eigenvalue problem. The governing equations and the boundary conditions for the problem is

$$-\frac{d^2u}{dx^2} = \lambda u(x)$$
$$\text{in } 0 \le x \le \pi \ \dots (7)$$

With Boundary Conditions, $u(0) = u(\pi) = 0$.

In equation (7), the λ is the eigenvalue, and u(x) is an eigen function. The eigenvalues are given by the squares of the integer numbers λ = 1, 4, 9, 16 . . . and that the eigenspaces corresponding to the eigenvalues are spanned by the eigenfunctions, sin (kx) for k = 1, 2, 3, 4 . . . . The B-spline Finite element method is used for the approximation of problem by considering the weak form of the governing equations that is given by

$$\int_0^\pi \frac{dN^T}{dx}\frac{dN}{dx}dx = \int_0^\pi N^T N dx \qquad \dots (8)$$

To obtain the approximate solution for the problem the domain is discretised with 8, 16, 32, and 64… nodes. All other parameters are identical with first test case - A Homogeneous Bar with Distributed Load. The eigenvalues obtained from the present method is compared with the analytical solution. The eigenvalues are tabled in Table (1) and found that the eigenfrequency is approaching to the exact values as the number of nodes is increasing.

Table 1: Eigen values computed for different values of n

Exact	n=8	n=16	n=32	n=64
1	1.0169	1.0037	1.0009	1.0002
4	4.2751	4.0588	4.0137	4.0033
9	10.4205	9.2998	9.0695	9.0168
16	20.4880	16.9559	16.2203	16.0531
25	35.1332	27.3567	25.5394	25.1298

3.3 Temperature distribution in a fin

Consider a heat transfer test case in a rectangular fin as shown in figure 5 (Lewis at al, 2004). The temperature distribution within rectangular fin is obtained by the B-Spline FEM treating it as one dimensional case. The governing differential equation for the fin problem is given by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - Ph(T - T_a) = 0, \quad 0 \le x \le L$$

The boundary conditions for this test case are,

$$T = T_i \quad \text{at } x = 0 \quad \text{and} \quad \frac{dT}{dx} = 0 \text{ at } x = L$$

In the above equations, T is temperature in the domain, h is the heat transfer coefficient, k is the thermal conductivity, A is the area of the cross section, P is the perimeter and T_a represents the ambient temperature.
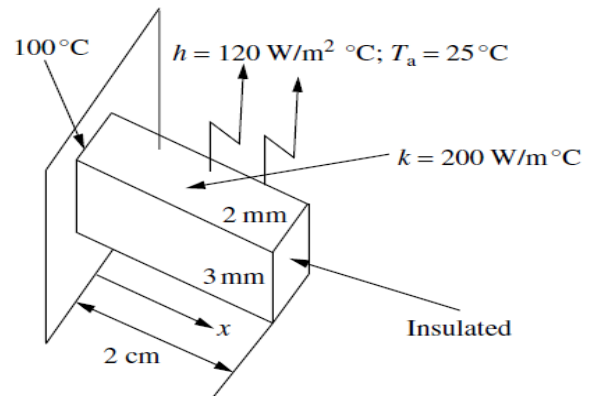

Fig 5. Heat Transfer in a Fin

The discretised equations based on the Galerkin approach for the fin problem is,

$$\left[kA\int_0^L \frac{dN^T}{dx}\frac{dN}{dx}dx + \int_0^L PhN^T N dx\right]\{T\} = PhT_a\int_0^L N^T dx$$

To obtain the approximate solution for the temperature distribution, the domain is discretised with 9 nodes and 8 elements. All other parameters are identical with first test case - A Homogeneous Bar with Distributed Load. The temperature obtained from the present method, as shown in figure 6, is compared with the exact solution given by the reference (Lewis at al, 2004).

4. Conclusion

In this work, an attempt is made to use the B-spline basis functions as the shape functions in the finite element method. An open uniform knot vector is used to obtain the

first degree B-Spline basis function. For the spatial discretization, the Galerkin's approximation method is employed. Two test cases have been performed to study the effectiveness of the current method. The results obtained by the present method are compared and found to be in good agreement with the analytical solution as well as the finite element method.
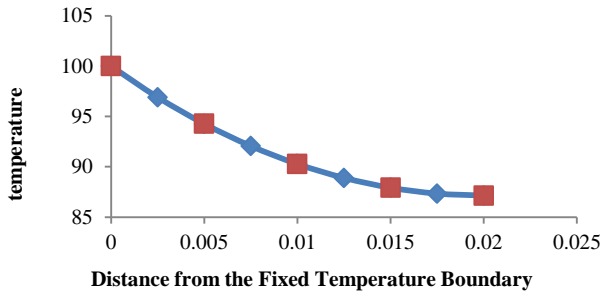

Fig.6: temperature distribution along the fin

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