International Journal of Current Engineering and Technology
ISSN 2277 – 4106

Available at http://inpressco.com/category/ijcet

Research Article

Hamming Index of Class of Graphs

Harishchandra S. Ramane[a*] and Asha B. Ganagi[a]

[a]Department of Mathematics, Gogte Institute of Technology, Udyambag, Belgaum - 590008, India

Abstract

Let A(G) be the adjacency matrix of a graph G. The rows of A(G) corresponding to a vertex v of G, denoted by s(v) is the string which belongs to Z_2^n, *a set of n-tuples. The Hamming distance between the vertices u and v is the number of positions in which s(u) and s(v) differ. The Hamming index of a graph G is the sum of the Hamming distances between all pairs of vertices of G. In this paper we obtain the Hamming index of certain class of graphs.*

Keywords: *Hamming distance, Hamming index. Adjacency matrix, Mathematics Subject Classification: 05C99*

1. Introduction

[1]Let $Z_2 = \{0, 1\}$ and $(Z_2\ +)$ be the additive group where + denotes addition modulo 2. For any positive integer n, $Z_2{}^n = Z_2 \text{ X } Z_2 \text{ X } \dots \text{ X } Z_2$ (n factors)
$= \{(x_1, x_2, \dots x_n \mid x_1, x_2, \dots x_n \in Z_2\ )\}$
Thus every element of $Z_2{}^n$ is an n-tuple $(x_1, x_2, \dots x_n\ )$ written as $x = x_1, x_2, \dots x_n$ where every x_i is either 0 or 1 and is called a string or word. The number of 1's in $x =_1, x_2, \dots x_n$ is called the *weight* of *x* and is denoned by wt(*x*).

Let $x = x_1, x_2, \dots x_n$ and $y = y_1, y_2, \dots y_n$ be the elements of $Z_2{}^n$. Then the sum $x + y$ is computed by adding the corresponding components of x and y under addition modulo2. That is $x_i + y_j = 0$ if $x_i = y_j$ and $x_i + y_j = 1$ if $x_i \neq y_j$

The *Hamming distance* $H_d(x, y)$ between the strings $x = x_1 x_2, \dots x_n$ and $y = y_1 y_2, \dots y_n$ is the number of i's such that $x_i \neq y_i$, $1 \leq i \leq n$.

Thus $H_d(x, y)$ = Number of positions in which x and y differ = wt(x+y)
Example: Let *x* = 01001 & *y* = 11010 and *x* + *y* = 10011,

Therefore $H_d(x, y)$ = wt(x+y) = 3

A graph *G* is called a *Hamming graph* [3,6] if each vertex $v \in V(G)$ can be labeled by a string of a fixed length s(v) such that $H_d(s(v), s(u)) = d_G(u, v)$ for all $u, v \in V(G)$ where $d_G(u, v)$ is the length of shortest path joining *u* and *v* in *G*.

Hamming graphs are known as an interesting graph family in connection with the error-correcting codes and association schemes. For more details one can refer [S. Bang et al,2008, V. Chepoi,1988, W. Imrich et al,1997, S. Klavzar et al, 2005, H. S. Ramane et al, 2013, R. Squer et al, 2001].

Motivated by the work on Hamming graphs in this we introduce and study the Hamming Index of a graph in association with its adjacency.

2. Preliminaries

Let G be an undirected graph without loops and multiple edges. Let $V(G) = \{v_1, v_2, \dots v_n\}$, $E(G) = \{e_1, e_2, \dots e_n\}$ be the vertex set and edge set of G respectively. The vertices adjacent to the vertex *v* are called the *neighbours* of *v*. The *degree* of a vertex v, denoted by *degv* is the number of neighbours of *v*.

The *adjacency matrix* of G is a square matrix $A(G) = [a_{ij}]$ of order *n*

where $a_{ij} = 1$ if the vertex v_i is adjacent to v_j

= 0 otherwise.

Rows of *A*(*G*) represents the strings which belongs to $Z_2{}^n$. The Row of *A* (*G*) corresponding to the vertex $v \in V(G)$is the string denoted by *s*(*v*).

Definition 2.1: *Hamming distance* between the vertices *u* and *v* of *G* denoted by $H_d(u, v)$ is defined as $H_d(u, v) = H_d(s(u), s(v))$

Definition 2.2: The *Hamming Index*[10] of a graph *G*, denoted by *H*(*G*) is defined as,

$$H(G) = \sum H_d(v_i, v_j) \quad 1 \leq i < j \leq n$$

*Corresponding author: **Harishchandra S. Ramane**

Hamming index is a graph invariant.

Ex. :

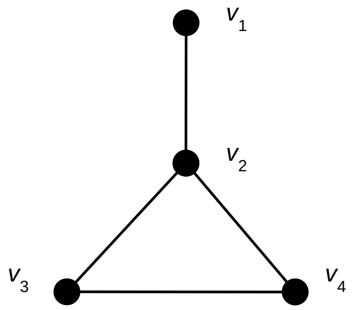

Fig. 1: Graph G

For a graph G of Fig. 1, the adjacency matrix is

$$A(G)=\begin{bmatrix}0&1&0&0\\1&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix}$$

and the strings are $s(v_1)=0100$,
$s(v_2)=1011$ $\quad s(v_3)=0101$ $\quad s(v_4)=0110$

$H_d(v_1,v_2)=4$ $\quad H_d(v_1,v_3)=1$ $\quad H_d(v_1,v_4)=1$

$H_d(v_2,v_3)=3$ $\quad H_d(v_2,v_4)=3$ $\quad H_d(v_3,v_4)=2$

Therefore $H(G)$ = 4+1+1+3+3+2 = 14

3. Hamming Index of certain class of graphs

Definition: Consider two graphs $G_1(n_1, m_1)$ and $G_2(n_2, m_2$. Then the graph whose vertex set $n_1 \mathrm{U} n_2$ and the edge set $m_1 \mathrm{U} m_2$ is called the union of G_1 and G_2 and is denoted by $G_1 \mathrm{U}\, G_2$.

Theorem 3.1: Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two graphs then Hamming index of union of G_1 and G_2 is,

$$H(G_1 \mathrm{U} \mathrm{G}_2)=H(\mathrm{G}_1)+H(\mathrm{G}_2)+2(n_1m_2+n_2m_1)$$

Proof:

Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two graphs then then the adjacency matrix of union of G_1 and G_2 is,

$$A\left(G_1 \mathrm{U} \mathrm{G}_2\right)=\begin{bmatrix}A(G_1) & O\\ O & A(\mathrm{G}_2)\end{bmatrix}$$

where $A(G_1)$ and $A(G_2)$ are adjacency matrices of G_1 and G_2 respectively, O is the null matrix.

Therefore,

$$H(G_1 \mathrm{U} \mathrm{G}_2)$$

$$=\sum_{1\le i<j\le n_1} H_{\mathrm{d}}(u_i,v_j)+\sum_{n_1\le i<j\le n_1+n_2} H_{\mathrm{d}}(u_i,v_j)+\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2} H_{\mathrm{d}}(u_i,v_j)$$

$$= H(G_1)+H(G_2)+\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2}\deg(u_i)+\deg(u_j)$$

$$=$$

$$H(G_1 \mathrm{U} \mathrm{G}_2)=H(\mathrm{G}_1)+H(\mathrm{G}_2)+2(n_1m_2+n_2m_1)$$

Definition: Consider two graphs $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$. The join of G_1 and G_2, denoted by $G_1 + G_2$, is $G_1 \mathrm{U} G_2$ along with each vertex of G_1 joined to every vertex of G_2 by an edge.

Theorem 3.2: Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two graphs then Hamming index of $G_1 + G_2$ is,

$$H(G_1+\mathrm{G}_2)=H(\mathrm{G}_1)+H(\mathrm{G}_2)+n_1n_2(n_1+n_2)-2n_1m_2-2n_2m_1$$

Proof:

Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two graphs then the adjacency matrix of join of G_1 and G_2 is,

$$A\left(G_1+\mathrm{G}_2\right)=\begin{bmatrix}\mathrm{A}(G_1) & J\\ J & \mathrm{A}(G_2)\end{bmatrix}$$

where $A(G_2)$ and $A(G_2)$ are adjacency matrices of G_1 and G_2 respectively and

$$J=\begin{bmatrix}1 & 1\ldots1\\ 1 & 1\ldots1\\ . & . \quad . \quad .\\ 1 & 1\ldots1\end{bmatrix}$$

Therefore,

$$H(G_1+\mathrm{G}_2)=\sum_{1\le i<j\le n_1+n_2} H_d(u_i,v_j)$$

$$=\sum_{1\le i<j\le n_1} H_{\mathrm{d}}(u_i,v_j)+\sum_{n_1\le i<j\le n_1+n_2} H_{\mathrm{d}}(u_i,v_j)+\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2} H_{\mathrm{d}}(u_i,v_j)$$

$$= H(G_1)+H(G_2)+\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2}\{n_1+n_2-[\deg(u_i)+\deg(u_j)]\}$$

$$\therefore \quad H(G_1+\mathrm{G}_2)=H(\mathrm{G}_1)+H(\mathrm{G}_2)+n_1n_2(n_1+n_2)-2n_1m_2-2n_2m_1$$

Definition: Let G be a graph with vertices $v_1, v_2, \ldots v_n$. A graph G^+ is obtained from G by inserting new vertices $u_1, u_2, \ldots u_n$, and joining u_i to v_i by an edge, i=1, 2, . . . n

Theorem 3.3: Let C_n *be cycle* on n vertices then Hamming index of C_n^+ is,

$$H(C_n^+) = \left[\frac{4n-5}{n-2}\right] H(C_n)$$

Proof:

Let C_n *be cycle* on n vertices then the adjacency matrix of C_n^+ is

$$A\left(C_n^+\right) = \begin{bmatrix} A(\mathrm{C_n}) & I \\ I & O \end{bmatrix}$$

where $A(C_n)$ is the adjacency matrix of C_n, I is the Identity matrix of order n and O is the null matrix

Therefore,

$$H(C_n^+) = \sum_{1\le i<j\le 2n} H_d(u_i, v_j)$$

$$= \sum_{1\le i<j\le n} H_{\mathrm{d}}(u_i, v_j) + \sum_{n+1\le i<j\le 2n} H_{\mathrm{d}}(u_i, v_j) + \sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j)$$

$$= \sum_{u,v\in C_n} 2 + H_{\mathrm{d}}(u,v) + \sum_{n+1\le i<j\le 2n} H_{\mathrm{d}}(u_i, v_j) + \sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) \quad \text{- - - - - (1)}$$

Where,

i)

$$\sum_{n+1\le i<j\le 2n} H_d(u_i, v_j) = 2({}^nC_2)$$

ii)

$$\sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) = \sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) \quad \text{for pair of } (u_i, v_j)\text{ adjacent pairs}$$
$$+ \sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) \quad \text{for pair of } (u_i, v_j)\text{ non-adjacent pairs}$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) = \text{hamming distances between n - adjacent pairs} = 4(\mathrm{n})$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) = \text{hamming distance between } (\mathrm{n}^2 - \mathrm{n}) \text{ non - adjacent pairs}$$

= Hamming distance between $2n$ pairs with common neighbour

+Hamming distance between $[(n^2\text{-n})\text{-}2n]$-pairs with non common neighbor= $2(2n) + 4(n^2\text{-}3\text{n})$

Substituting in eqn(1),

$$H(C_n^+) =$$
$$\left[H(C_n) + 2\,{}^nC_2\right] + 2\,{}^nC_2 + \left[4n + 2(2n) + 4(n^2 - 3n)\right]$$
$$= H(C_n) + 6n^2 - 6n$$
$$= 2n(n-2) + 6n^2 - 6n$$

[H. S. Ramane et al, 2013]

= 2n(4n-5

$$= \left[\frac{4n-5}{n-2}\right] H(C_n)$$

Theorem 3.4: Let K_n *be complete graph* on n vertices then Hamming index of K_n^+ is,

$$H(K_n^+) = H(K_n) + n^3 + n^2$$

Proof:

Let K_n *be complete graph* on n vertices then the adjacency matrix of K_n^+ is

$$A\left(K_n^+\right) = \begin{bmatrix} A(\mathrm{K_n}) & I \\ I & O \end{bmatrix}$$

where $A(\mathrm{K_n})$ is the adjacency matrix of $\mathrm{K_n}$, I is the Identity matrix of order n and O is the null matrix

Therefore,

$$H(K_n^+) = \sum_{1\le i<j\le 2n} H_d(u_i, v_j)$$

$$= \sum_{1\le i<j\le n} H_{\mathrm{d}}(u_i, v_j) + \sum_{n+1\le i<j\le 2n} H_{\mathrm{d}}(u_i, v_j) + \sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j)$$

$$= \sum_{u,v\in C_n} 2 + H_{\mathrm{d}}(u,v) + \sum_{n+1\le i<j\le 2n} H_{\mathrm{d}}(u_i, v_j) + \sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) \quad \text{- - - - - (1)}$$

Where,

i)

$$\sum_{n+1\le i<j\le 2n} H_d(u_i, v_j) = 2({}^nC_2)$$

ii)

$$\sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) = \sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) \quad \text{for pair of } (u_i, v_j)\text{ adjacent pairs}$$
$$+ \sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) \quad \text{for pair of } (u_i, v_j)\text{ non-adjacent pairs}$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_{\mathrm{d}}(u_i, v_j) = \text{hamming distances between n - adjacent pairs} = (\mathrm{n}+1)(\mathrm{n})$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_d(u_i, v_j) = \text{hamming distance between } (n^2 - n) \text{ non-adjacent pairs} = (n-1)(n^2-n)$$

Substituting in eqn(1),

$$H(K_n^+) =$$
$$\left[H(K_n) + 2\,{}^nC_2\right] + 2\,{}^nC_2 + \left[n(n+1) + (n-1)(n^2-n)\right]$$
$$= H(K_n) + n^3 + n^2$$

Acknowledgement

Authors are thankful to Prof. P. R. Hampiholi for his suggestions.

References

S. Bang, E. R. van Dam, J. H. Koolen (2008), Spectral characterization of the Hamming graphs, *Linear Algebra Appl.*, 429, 2678–2686

V. Chepoi (1988), d-connectivity and isometric subgraphs of Hamming graphs, *Cybernetics,* 1 (1988), 6–9.

W. Imrich, S. Klavˇzar (1993), A simple O (mn) algorithm for recognizing Hamming graphs, Bull. Inst. Combin. Appl., 9, 45–56.

W. Imrich, S. Klavˇzar (1996), On the complexity of recognizing Hamming graphs and related classes of graphs, *European J. Combin.*, 17, 209–221.

W. Imrich, S. Klavˇzar, Recognizing Hamming graphs in linear time and space, *Inform. Process. Lett.*, 63 (1997), 91–95.

S. Klavˇzar, I. Peterin (2005), Characterizing subgraphs of Hamming graphs, J. Graph Theory, 49, 302–312.

B. Kolman, R. Busby, S. C. Ross (2002), DiscreteMathematical Structures, Prentice Hall of India, New Delhi,

M. Mollard (1991), Two characterizations of generalized hypercubes, Discrete Math., 93, 63–74.

B. Park, Y. Sano (2011), The competition numbers of Hamming graphs with diameter at most three, J. Korean Math. Soc., 4, 691–702.

Harishchandra S. Ramane and Asha B. Ganagi (2013), Hamming distance between vertices and Hamming index of graphs, SIAM Journal on Discrete Mathematic(preprint)

R. Squier, B. Torrence (2001), A. Vogt, The number of edges in a subgraph of a Hamming graph, Appl. Math. Lett., 14,701–705

D. B. West (2009), Introduction to Graph Theory, PHI Learning, New Delhi