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Research Article

Homotopy Perturbation Method for Solving Linear Boundary Value Problems

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Abstract

A homotopy perturbation method (HPM) is introduced for obtaining solutions of systems of linear partial differential equations. The non-Newtonian fluid under consideration is obeying the Casson model. Heat and mass transfer for unsteady flow with heat sources and the effect of chemical reaction are studied. The main advantage of this method is the flexibility to give approximate and exact solutions to both linear and nonlinear problems. The results show that the methods are simple and effective. The distributions of the velocity, temperature and concentration functions are discussed and illustrated graphically for different values of the physical parameters of the problem.

Keywords: *HPM, Casson model etc.*

Introduction

The theory and application of optimal control are relevant to fields such as biomedicine, physics, economy, aerospace, chemical engineering, and robotics. However, the optimal control of nonlinear systems is a difficult task, which has been studied for decades. Identification of the thermal conductivity coefficient and heat source is important in engineering practice. Both problems belong to the class of inverse problems and can be solved by means of homotopy perturbation method. The homotopy perturbation method (HPM) has been worked out over a number of years by numerous authors, mainly by He J. In (HPM), a complicated problem under study is continuously deformed into a simple problem which is easy to solve to obtain an analytic or approximate solution investigated by He J. Demir A *et al*. explained that a new homotopy perturbation technique is proposed to find a solution based on the decomposition the function f (x, y) which leads to the construction of new homotopies and clarify that the proposed method (HPM) is very effective and simple. Khyati R. *et al*. Investigate the exact solution of linear and nonlinear diffusion equations are obtained by Homotopy Perturbation Method. a hybrid method that combines the homotopy perturbation method (HPM) and Padétechnique to obtain the approximate analytic solution of the Hamilton–Jacobi–Bellman equation is Investigated by Ganjefar S. and Rezaei S. Babolian E.*et. al.* use the homotopy perturbation method for solving time-dependent differential equations. Patra A. and Saha S. enhance the heat transfer between primary surface and the environment, radiating extended surfaces are commonly utilised. Temperature distribution and effectiveness of convective radial fins with constant and temperature-dependent thermal conductivity are solved by applying homotopy perturbation sumudu transform method (HPSTM). The homotopy perturbation method (HPM) combined with Trefftz method is employed to find the solution of two kinds of nonlinear inverse problems for heat conduction investigated by Grysa K. and Maciag A. Tsai C. investigated the homotopy analysis method (HAM) is combined with the method of fundamental solutions (MFS) and the augmented polyharmonic spline (APS) to solve certain nonlinear partial differential equations (PDE).

The purpose of the present investigation is to study the homotopy perturbation method for solving linear deferential equation which flow of couple-stress. We restrict ourselves to the case of linear differential equations and suggest a quite simple technique for using HPM. Comparatively speaking, even though we don't claim that the suggested technique is the best one, this is a reliable technique which one can simply use without having much experience and understanding of HPM.

Mathematical formulation and solution

Channel flow between two oscillating porous plates $y = 0$ and $y = h$, the fluid is being injected by one plate with constant velocity V and sucked off by the other plate with the same velocity. Then the continuity equation reduces to $\frac{\partial u}{\partial x} = 0$ so that u is the function of y and t only. A uniform magnetic field

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with magnetic flux density vector $B = (0, B_0, 0)$ is applied. Also the heat and mass transfer in the channel are taken into account by giving temperature and concentration to the lower and upper plate as T_0, C_0 and T_w, C_w respectively.

A two dimensional a Cartesian coordinates $(x\ , y)$ are characterized and considered, where x is in the direction of fluid motion and y is perpendicular to the mean position of the channel.

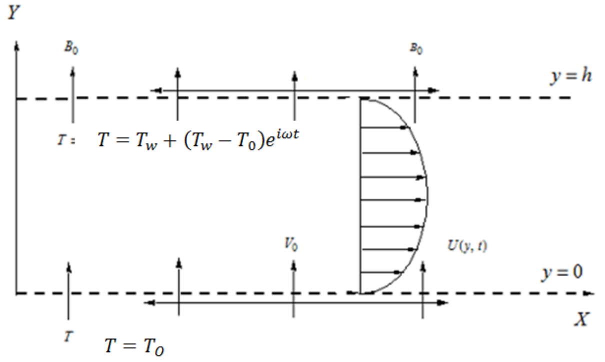

Fig.1: Geometry of the problem

he constitutive equation of the Bi-viscosity model can be written in the following tensor notation as:

$$\tau_{ij} = 2\left(\mu_B + P_y\sqrt{2\pi c}\right)e_{ij}\ , e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Where μ_B is the plastic viscosity, P_y is the yielding stress, $\pi = e_{ij}e_{ij}$, which e_{ij} is the i, j component of the deformation rate and the value of β denotes the upper limit of apparent viscosity coefficient. For Newtonian fluid. $P_y = 0$ Where u, V_0 are the for unsteady two-dimensional flows, the velocity, temperature and concentration can be written as a function of y *and* t.

The viscoelastic fluid in x-direction is driven by a pulsatile pressure gradient

$$\frac{\partial p^*}{\partial x} = A^* + B^* e^{i\omega t}$$

Where A^* and B^* are known quantities represents the steady-state of the pressure gradient and oscillatory parts respectively where ω is the frequency of pulsatile flow.

Basic equations

The equations governing the flow of an incompressible viscoelastic fluid are given by:

The continuity equation

$$\nabla.\underline{V} = 0 \tag{1}$$

The momentum equation

$$\rho\left[\frac{\partial V}{\partial t} + (\underline{V}.\nabla)V\right] = -\nabla P + \nabla.\underline{\tau} + \mu_e\left(\underline{J}\wedge\underline{B}\right) - \eta\nabla^4\underline{V} \tag{2}$$

The temperature equation with radiation

$$c_p\rho\left[\frac{\partial T}{\partial t} + (\underline{V}.\nabla)T\right] = k_0\nabla^2 T - \mathrm{Q}\,\mathrm{T} \tag{3}$$

The concentration equation with diffusion and chemical reaction

$$\left[\frac{\partial C}{\partial t} + (\underline{V}.\nabla)C\right] = D_m\nabla^2 C - \zeta C \tag{4}$$

Where $\underline{V}$ is the velocity, τ is the extra stress tensor obeying Casson model, ρ is the liquid density, P is the pressure, μ is the viscosity of the fluid. μ_e permeability of magnetic field, $\underline{J}$ is current density. T and C temperature and concentration of the fluid, D_m is the coefficient of mass diffusivity and c_p is the specific heat capacity at constant pressure , K is the mean absorption coefficient, T_m is the mean temperature , Q is heat flux, ζ is constants of chemical reaction. Thus, for the effect of couple stress to be present $V_{i,rrss}$ must be nonzero.

For unsteady two-dimensional flow, the velocity, temperature and concentration can be written as a function of y and t only.

$$V = (u(y,t), V_0). \quad , T = T(y,t), C = C(y,t).$$

where u *and* V_0 be the longitudinal and transverse velocity components of the fluid, respectively.

Equations (1-4) reduce to

$$\frac{\partial u}{\partial x} = 0 \tag{5}$$

$$\frac{\partial u}{\partial t} + V_0\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu_B(1 + \beta^{-1})\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma B_0^2}{\rho}u - \eta\left(\frac{\partial^4 u}{\partial y^4}\right) \tag{6}$$

$$\frac{\partial T}{\partial t} + V_0\frac{\partial T}{\partial y} = \frac{k_0}{c_p\rho}\left(\frac{\partial^2 T}{\partial y^2}\right) - \mathrm{Q}\,\mathrm{T} \tag{7}$$

$$\frac{\partial C}{\partial t} + V_0\frac{\partial C}{\partial y} = D_m\left(\frac{\partial^2 C}{\partial y^2}\right) - \zeta\, C \tag{8}$$

In mathematical form, the corresponding boundary conditions can be put as

$$\left.\begin{array}{l} u = V_0 e^{iwt},\ T = T_w + e^{iwt}(T_w - T_0)\ C = C_w + e^{iwt}(C_w - C_0)\ at\ y = h \\ u = V_0 e^{iwt},\ T = T_0 \qquad C = C_0 \qquad at\ y = 0 \end{array}\right\} \tag{9}$$

Let us introduce the following dimensionless quantities as follow:

$$\left.\begin{array}{lll} y^* = \frac{y}{h} & u^* = \frac{u}{V_0}, & t^* = \frac{t\,V_0}{h} \\ R_e = \frac{V_0 h}{v}, & p^* = \frac{hp}{\rho v V_0} & T^* = \frac{T - T_0}{T_w - T_0}, \\ c^* = \frac{C - C_0}{C_w - C_0}, & k_p = \frac{k_p{}^*}{\rho\, h^2}\ ,, & M^2 = \frac{\sigma\, h^2 B_0{}^2}{\rho v}\ , \\ S_c = \frac{v}{D_m} & \omega^* = \frac{\omega\, h}{V_0} & p_r = \frac{V_0 c_p h\,\rho}{k_0} \end{array}\right\} \tag{10}$$

Where M^2 is the magnetic field parameter, S_c is the Schmidt number, and E_c the Eckhart number, R_e is the Reynolds number, p_r is the Prandtle number .

The system of equations (5-8) with the condition (10) can be written in dimensionless form after dropping star mark as follows:

$$R_e\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{1}{R_e}(1+\beta^{-1})\left(\frac{\partial^2 u}{\partial y^2}\right)-M^2 u-\frac{1}{R_c}\left(\frac{\partial^4 u}{\partial y^4}\right) \qquad (11)$$

$$R_e p_r\left(\frac{\partial T}{\partial t}+\frac{\partial T}{\partial y}\right)=\frac{\partial^2 T}{\partial y^2}-\gamma\, T \qquad (12)$$

$$R_e\left(\frac{\partial \mathrm{C}}{\partial \mathrm{t}}+\frac{\partial \mathrm{C}}{\partial \mathrm{y}}\right)=\frac{1}{S_c}\left(\frac{\partial^2 \mathrm{C}}{\partial \mathrm{y}^2}\right)-\xi C \qquad (13)$$

Where R_c is the couple stress parameter and R_n is the radiation parameter.

The corresponding dimensionless boundary conditions are:

$$\left.\begin{array}{llll} u=e^{i\omega t}, & T=1+e^{i\omega t} & C=1+e^{i\omega t} & at\ \ y=1 \\ u=e^{i\omega t}, & T=0 & C=0 & at\ \ y=0 \end{array}\right\} \qquad (14)$$

Solution of the problem

To solve equations (11-13) subjected to the boundary conditions (14), we write the velocity, temperature and concentration in the form:

$$u=u_s(y)+u_f(y)e^{i\omega t} \qquad (15)$$
$$T=T_s(y)+T_f(y)e^{i\omega t} \qquad (16)$$
$$C=C_s(y)+C_f(y)e^{i\omega t} \qquad (17)$$

Equations ((11)-(13)) take the following form:

Steady case

$$-\frac{1}{R_c}u_s^{\backslash\backslash\backslash\backslash}+\alpha_2 u_s^{\backslash\backslash}+R_e u_s^{\backslash}-M^2 u_s=A \qquad (18)$$

$$T_s^{\backslash\backslash}-R_e p_r T_s^{\backslash}-\gamma\, T_s=0 \qquad (19)$$

$$\frac{1}{S_c}\mathrm{C}_s^{\backslash\backslash}-\mathrm{C}_s^{\backslash}-\xi\, C=0 \qquad (20)$$

Unsteady case

$$-\frac{1}{R_c}u_f^{\backslash\backslash\backslash\backslash}+\alpha_1 u_f^{\backslash\backslash}-R_e u_f^{\backslash}-(M^2+i\omega R_e)u_f=B \qquad (21)$$

$$T_f^{\backslash\backslash}-R_e P_r T_f^{\backslash}-(\gamma+i\omega R_e P_r)\,T_f=0 \qquad (22)$$

$$\frac{1}{S_c}\mathrm{C}_f^{\backslash\backslash}-\mathrm{C}_f^{\backslash}-\xi\,\mathrm{C}_f=0 \qquad (23)$$

Equations ((18)-(23)) can be solved by using the following perturbation method for small homotopy perturbation parameter (q),

$$u_s(y)=u_{s0}(y)+q\,u_{s1}(y)+O(q) \qquad (24)$$

$$u_f(y)=u_{f0}(y)+q\,u_{f1}(y)+O\,(q) \qquad (25)$$

Collecting the coefficients of like power of (q), we get the following set of equations of motion:

$$\alpha_2 u_{so}{}^{\backslash\backslash}-R_e u_{so}{}^{\backslash}-\alpha_1 u_{so}=A \qquad (26)$$

$$\alpha_2 u_{s1}{}^{\backslash\backslash}-R_e u_{s1}{}^{\backslash}-M^2 u_{s1}=-u_{s0}{}^{\backslash\backslash\backslash\backslash} \qquad (27)$$

$$\alpha_2 u_{f0}{}^{\backslash\backslash}-R_e u_{f0}{}^{\backslash}-(M^2+i\omega R_e)u_{f0}=B \qquad (28)$$

$$\alpha_2 u_{f1}{}^{\backslash\backslash}-R_e u_{f1}{}^{\backslash}-(M^2+i\omega R_e)u_{f1}=-u_{f0}{}^{\backslash\backslash\backslash\backslash} \qquad (29)$$

While the boundary conditions will take the following form:

$$\left.\begin{array}{lllll} u_{s0}=0\,, u_{s1}=0\,, & T_s=1, & C_s=1, & at\ \ y=1 \\ u_{s0}=0\,, \ u_{s1}=0\,, & T_s=0, & C_s=0, & at\ \ y=0 \\ u_{f0}=1\,, u_{f1}=0\,, & T_f=1, & C_f=1, & at\ \ y=1 \\ u_{f0}=1\,, u_{f1}=0\,, & T_f=0, & C_f=0, & at\ \ y=0 \end{array}\right\} \qquad (30)$$

Hence, the solutions of equations ((18)-(23)) subject to boundary conditions (30) can be obtained as follows respectively:

Steady case

$$u_{s0}=\mathrm{c}_1 e^{m_1 y}+\mathrm{c}_2 e^{m_2 y}-\frac{A}{M^2} \qquad (31)$$

$$u_{s1}=(\mathrm{c}_3+\beta_1)e^{m_1 y}+(\mathrm{c}_4+\beta_2)e^{m_2 y} \qquad (32)$$

$$T_s=\mathrm{c}_9 e^{m_9 y}+\mathrm{c}_{10}e^{m_{10} y} \qquad (33)$$

$$C_s=\mathrm{c}_{13}e^{m_{13} y}+\mathrm{c}_{14}e^{m_{14} y}+\beta_5 e^{m_9 y}+\beta_6 e^{m_{10} y} \qquad (34)$$

Unsteady case

$$u_{f0}=c_5\,e^{m_7 y}+c_6 e^{m_8 y} \qquad (35)$$

$$u_{f1}=(\mathrm{c}_7+\beta_3)e^{m_7 y}+(\mathrm{c}_8+\beta_4)e^{m_8 y} \qquad (36)$$

$$T_f=\mathrm{c}_{11}e^{m_{11} y}+\mathrm{c}_{12}e^{m_{12} y} \qquad (37)$$

$$C_f=\mathrm{c}_{15}e^{m_{15} y}+\mathrm{c}_{16}e^{m_{16} y}+\beta_7 e^{m_9 y}+\beta_8 e^{m_{10} y} \qquad (38)$$

Results and discussion

In the present work we applied the homotopy perturbation method to solve linear differential equations. The problem of an incompressible unsteady (MHD) pulsatile flow of couple-stress non-Newtonian fluid obeying Casson model with heat and mass transfer in two-dimensional channel has been studied.

We discussed the influences of the magnetic field and porosity of the medium on the fluid. The formula of the velocity, temperature and concentration have been obtained. Different parameters of the problem have been showed graphically such as the magnetic parameter M^2, the couple stress parameter R_c ,the parameter of non-Newtonian fluid β , Prandtl number P_r, and S_c Shmidt number and the chemical reaction ξ. We compare the results with other researches solved with other ways.

Figure (2) illustrates the change of the velocity distribution u versus the coordinate y with several value of R_e , In this figure the velocity increases when the Reynolds number R_e increases. The relation between velocity distribution u and couple stress parameter R_c has been presented in figure (3) where the velocity distributions decreases with the increasing of R_c. Figure (4) illustrates the change of the velocity distribution u versus the coordinate y with several value of M^2. in this figure velocity distribution u increases with the increasing of the magnetic field parameter M^2. We can observe from figure (5) that the velocity u increases with the increases of R_e according to the time variation. It is found that the velocity distribution increases (decreases) with the increases of R_c according to the time variation in figure (6). in figures (7) the velocity distribution increases with the increases of M according to the time variation.

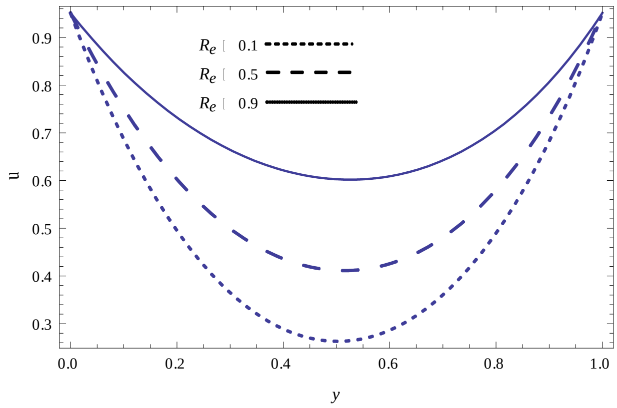

Figure 2: The velocity distribution u is plotted with the coordinates y for different values of Reynolds number R_e

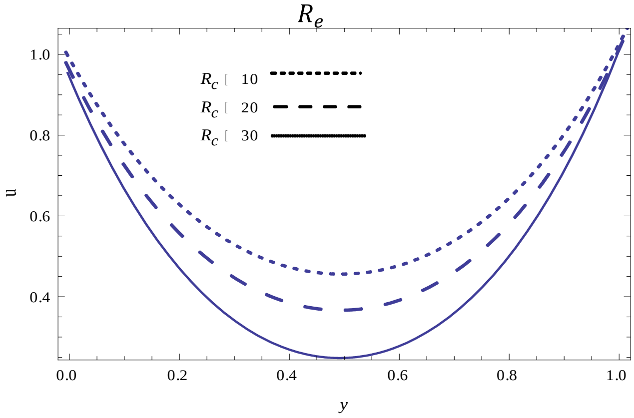

Figure 3: The velocity distribution u is plotted with the coordinates y for different values of couple stress parameter R_c

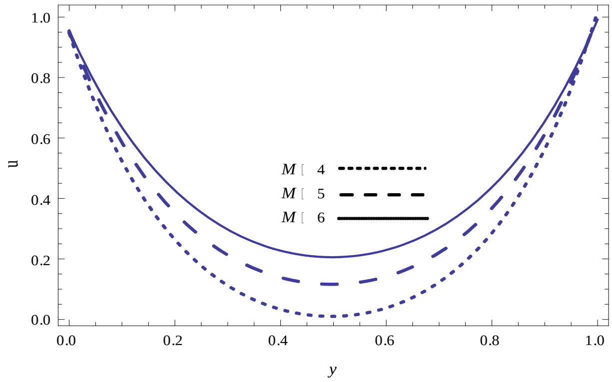

Figure 4: The velocity distribution u is plotted with the coordinates y for different values of magnetic field parameter M^2

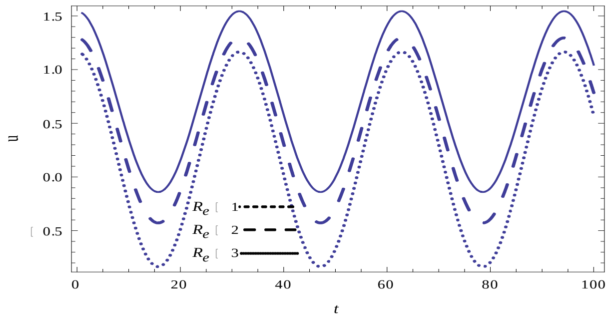

Figure 5: The velocity distribution u is plotted with the time t for different values of Reynolds number R_e

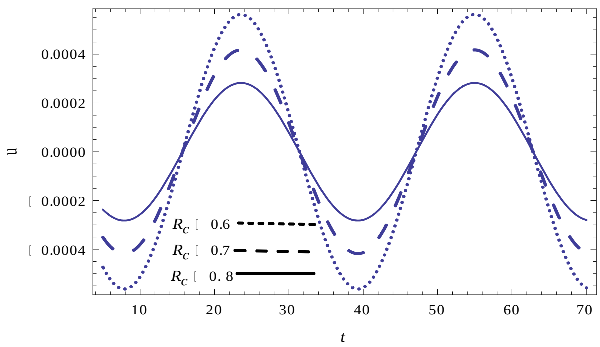

Figure 6: The velocity distribution u is plotted with the coordinates y for different values of couple stress parameter R_c

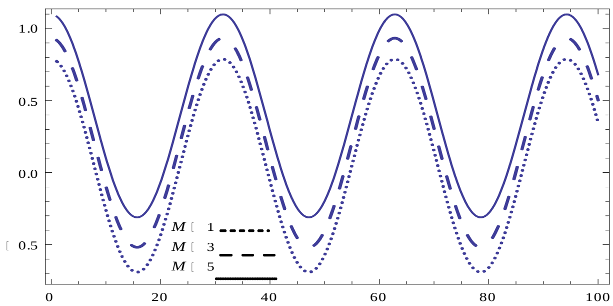

Figure 7: The velocity distribution u is plotted with the time t for different values of magnetic field parameter M^2

Conclusion

In this work, we used homotopy perturbation method for solving linear differential equations. The results have been approved the efficiency of this method for solving this problem. Furthermore, accurate solutions were derived from first-order approximations in the examples presented in this paper. To clarify the influences of the various parameters of the problem at hand, a numerical calculating is made. The concluding remarks may be listed as follows.

1) The velocity distribution u of the fluid has the same behavior with M and Reynolds number R_e . It increases with the increase of M and R_e **.** but the opposite with R_c.
2) The results which obtained in the steady state for boundary conditions agreed with both of Das U. and Adhikary S. and Misra J. where the velocity distribution u of the fluid in the steady state has the same behavior with magnetic field parameter M .

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