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Research Article

Dynamic Nodal Connectivity in Finite Element Method using Bezier Basis Functions

CH Sridher Reddy[#], A Mamatha Reddy[$] and Ram Kishore Vodnala[$*]

[#]Department of Mechanical Engineering, JNTUH College of Engineering Manthani, Karimnagar, India
[$]Department of Mechanical Engineering, JNTUH College of Engineering Jagitial, Karimnagar, India

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Abstract

In this work an attempt is made to define the nodal connectivity at the runtime rather than defining the same at the initial stage. The shape functions are also need to be defined dynamically. The advantage of this is in the selection of degree of the shape function as required by the gradient of the field variable. If the shape functions are defined by the Bezier basis functions, the nodal connectivity can be obtained at the run time itself. Apart from this, all other procedure like the development of the stiffness matrix, force vector and the assembly and the solution stage are identical to the Finite Element method. The results obtained by the present method are compared and found to be in good agreement with the analytical solution and the finite element method.

Keywords: *Dynamic nodal connectivity, Bezier basis functions, Homogeneous bar problem , Eigen value problem, Heat transfer in Fin problem.*

1. Introduction

The finite element method is comprehensively developed and is used to analyze various physical problems in the field of solids, structures, heat transfer, and fluid flow and in many other fields with arbitrary shape, loads and support conditions. In this method, the problem domain is divided into sub-domains called elements which are geometrically defined by predefined connectivity between the nodal points. These nodal points carry the field variable as nodal unknowns. Within each element, the variation of the field variable is obtained from the nodal values through the shape functions.

However, this Finite element method also has some shortcomings like low accuracy of stress field, time consuming mesh generation, adaptivity and re-meshing at high gradient field variable. The root cause of all these are due to the fixed nodal connectivity. Once the meshing has been done, the nodal connectivity needs to be changed either for fine mesh of the same element by increasing number of elements and nodes in the domain or for higher order elements as by increasing the number of nodes per each element. In both the cases, the re-meshing needs to be carried out.

In the present work, an attempt is made to define the nodal connectivity at the runtime rather than defining the same at the initial stage. The shape functions are also defined dynamically. Numerical studies are performed with problems of linear elasticity and Heat Conduction in one dimension. The problems considered are the extension of a prismatic bar due to body load, 1D Laplace eigenvalue problem and temperature distribution in a rectangular fin.

2. Dynamic Nodal Connectivity

Defining the nodal connectivity at the runtime rather than defining the same at the initial stage. The shape functions are also need to be defined dynamically. The advantage of this is in the selection of degree of the shape function as required by the gradient of the field variable.

3. Bezier Basis Functions

A Bezier curve is a parametric curve that uses the Bernstein polynomials as a basis. A Bezier curve of n^{th} degree (i.e., n+1 order) is represented as,

$$P(t) = \sum_{i=0}^{n} B_{n,i}(t)P_i$$

$$B_i = n_{c_i}\left(\frac{t_1-t}{t_1-t_0}\right)^{n-i}\left(\frac{t-t_0}{t_1-t_0}\right)^{i} \quad (1)$$

*Corresponding author: **Ram Kishore Vodnala**

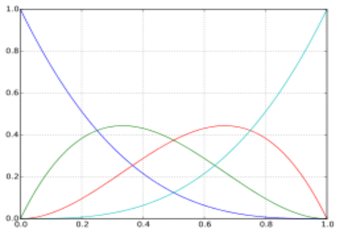

Fig.1 The basis functions on the range t in [0,1] for cubic Bézier curves: blue: $B_{3,0} = (1 - t)^3$, green: $B_{3,1} = 3(1 - t)^2 t$, red: $B_{3,2} = 3(1 - t) t^2$, and cyan: $B_{3,3} = t^3$.

3.1 Bezier basis functions as Shape functions in FEM

There are two requirements for a shape function. One is sum of all shape functions should be equal to 1 ($\sum N_i = 1$) & It must take a unit value at node i, and is zero at all other nodes (N_i= 1 at node i ; N_i=0 at all other nodes).

The above two requirements will be conveniently satisfied by taking Bezier basis functions as shape functions. For any given value of the parameter t, the summation of the basis functions is exactly equal to 1. i.e.,
$\sum_{i=0}^{n} B_i=1$ & $B_i=1$ *at i and* $B_i=0$ at other points.

Hence, In this work Bezier basis functions are considered as shape functions.

4. The Test Problems

Four test problems are considered to study the effectiveness of the present method. They are a homogeneous bar subjected to distributed loads acting along the longitudinal axis, one dimensional eigenvalue problem of finding the frequencies of longitudinal vibration of a rod and temperature distribution in a rectangular fin.

4.1 A Homogeneous Bar with Linear Distributed Load

A homogeneous bar of length L, which is subjected to a linear distributed body load 'f = x', is considered as a first test case. The mathematical model for the physical system shown in figure (2) is represented by one dimensional Poisson equation that is given by,

$$\frac{d^2u}{dx^2} + x = 0 \ , 0 \le x \le 1 \qquad (2)$$

With boundary conditions, $u(0) = 0 \,\&\, u'(1) = 0$,

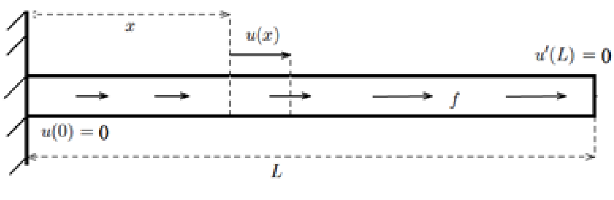

Fig.2 Homogeneous Bar with Distributed Load

The weak form for the governing equation (2) is obtained by using weigh function W,

$$\int_0^1 W\left(\frac{d^2u}{dx^2} + x\right)dx = 0 \qquad (3)$$

In the present method, U can be written as a linear combination of Bezier basis functions (Bernstein Polynomials) and the nodal parameters of deformation as,

$$U=\sum_{i=0}^{n} N_i\, u_i \qquad (4)$$

Where N_i are the basis functions of degree n as given by

$$N_i=n_{c_i}\left(\frac{t_1-t}{t_1-t_0}\right)^{n-i}\left(\frac{t-t_0}{t_1-t_0}\right)^{i}$$

Integrating by parts and using the Bezier basis function (Equation 3) and the boundary conditions, linear system of equations are obtained as **KU=F**

where,

$$\text{K} = \left[\int_0^1 \frac{dN^T}{dx}\frac{dN}{dx}dx\right]; \quad \text{U} = \{u\}; \quad \text{F} = \left[\int_0^1 xN^T dx\right] \qquad (5)$$

Where N represents the Bezier basis functions. The stiffness matrix (K) and the force vector (F) are evaluated by using two point gauss quadrature.

$$\mathbf{K} = \left[\int_0^1 \frac{dN^T}{dx}\frac{dN}{dx}dx\right]; \quad \mathbf{U} = \{u\}; \quad \mathbf{F} = \left[\int_0^1 xN^T dx\right] \qquad (6)$$

The domain is discretized with 21 nodes and 3 number of control points/nodes at a time for the nodal connectivity. And obtained results are compared with the exact solution given by

$$u_{ex}(x) = \frac{x}{2} - \frac{x^3}{6} \quad \text{........ } (7)$$

The figure (3) shows the displacement field along the length of the bar. It can be observed from the figure(3) that the results obtained by the present method are in very good agreement with the exact solutions.

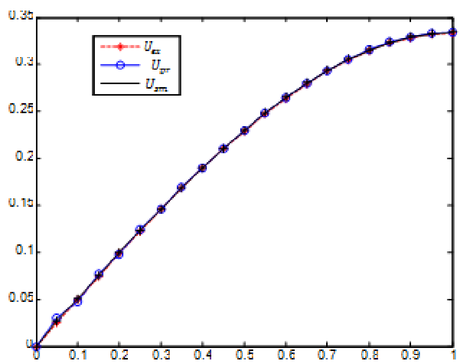

Fig .3 Comparison of field variable with exact solution

4.2 A Homogeneous Bar with Quadratic Distributed Load

A homogeneous bar of length L, which is subjected to a quadratic distributed body load 'f = **x**', is considered Governing equation is

$$\frac{d^2u}{dx^2} + x^2 = 0 \tag{8}$$

Where *u* represents the deformation in the homogeneous bar.
With boundary conditions u(0) = u(1) = 0 having Exact solution shown in equation (9)

$$U_{ex}(x) = \frac{x}{3} - \frac{x^4}{12} \tag{9}$$

The weak form for the governing equation is obtained by using the Bezier basis function as weight function

$$\int_0^1 w\left(\frac{d^2u}{dx^2} + x^2\right)dx = 0 \tag{10}$$

Where w is the weight function Integrating by parts and using the basis functions (Equation 10) and the boundary conditions, a linear system of equations are obtained as **KU** = **F** where,

$$K = \left[\int_0^1 \frac{dN^T}{dx}\frac{dN}{dx}dx\right]; \qquad U = \{u\};$$

$$F = \int_0^1 [N]^T x^2 dx$$

The domain is discretized with 21 nodes with a coordinate space of 1 and 2 number of nodes are taken . The obtained results are compared with the exact solution are given below.

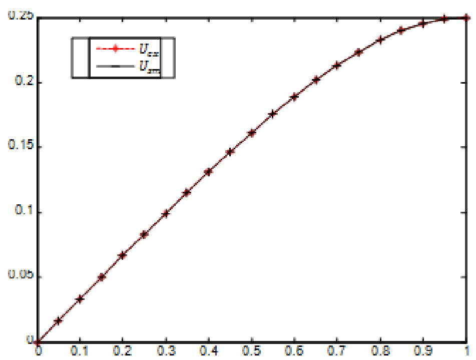

Fig.4 Comparison of field variable with exact solution

4.3 A one-dimensional eigen value problem

In this section, the current method is applied to a 1D Laplace eigenvalue problem. The governing equations and the boundary conditions for the problem is

$$-\frac{d^2u}{dx^2} = \lambda u(x) \quad \text{in } 0 \le x \le \pi \tag{11}$$

With Boundary Conditions, $u(0) = u(\pi) = 0$.

In equation (11), the λ is the eigenvalue, and u(x) is an eigen function. The eigenvalues are given by the squares of the integer numbers λ = 1, 4, 9, 16 . . . and that the eigen spaces corresponding to the eigenvalues are spanned by the eigen functions, sin (kx) for k = 1, 2, 3, 4 . . . . The Dynamic nodal connectivity in Finite element method is used for the approximation of problem by considering the weak form of the governing equations that is given by

$$\int_0^\pi \frac{dN^T}{dx}\frac{dN}{dx}dx = \int_0^\pi N^T N dx \tag{12}$$

To obtain the approximate solution for the problem the domain is discretised with 8, 16, 32, and 64… nodes. All other parameters are identical with first test case - A Homogeneous Bar with Linear Distributed Load. The eigenvalues obtained from the present method is compared with the analytical solution. The eigenvalues are tabled in Table (1) and found that the eigen frequency is approaching to the exact values as the number of nodes is increasing.

Table 1 : Eigen values computed for different values of n

Exact	n=8	n=16	n=32	n=64
1	1.0169	1.0037	1.0009	1.0002
4	4.2751	4.0588	4.0137	4.0033
9	10.4205	9.2998	9.0695	9.0168
16	20.4880	16.9559	16.2203	16.0531
25	35.1332	27.3567	25.5394	25.1298

4.4 Temperature distribution in a fin

Consider a heat transfer test case in a rectangular fin as shown in figure (5) (Lewis at al, 2004). The temperature distribution within rectangular fin is obtained by the Dynamic nodal connectivity in FEM treating it as one dimensional case. The governing differential equation for the fin problem is given by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - Ph(T - T_a) = 0, \quad 0 \le x \le L \tag{13}$$

The boundary conditions for this test case are,

$$T = T_i \quad \text{at} \quad x = 0 \quad \text{and} \quad \frac{dT}{dx} = 0 \quad \text{at} \quad x = L$$

In the above equations, T is temperature in the domain, h is the heat transfer coefficient, k is the thermal conductivity, A is the area of the cross section, P is the perimeter and T_a represents the ambient temperature.

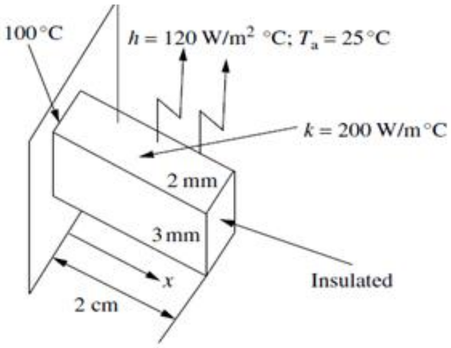

Fig.5 Heat transfer in fin

The discretised equations based on the Galerkin approach for the fin problem is,

$$\left[kA\int_0^L \frac{dN^T}{dx}\frac{dN}{dx}dx + \int_0^L PhN^T N dx \right]\{T\} = PhT_a \int_0^L N^T dx \qquad (14)$$

To obtain the approximate solution for the temperature distribution, the domain is discretised with 21 nodes. All other parameters are identical with first test case - A Homogeneous Bar with Linear Distributed Load. The temperature obtained from the present method, as shown in figure (6), is compared with the exact solution given by the reference (Lewis *et al,* 2004).

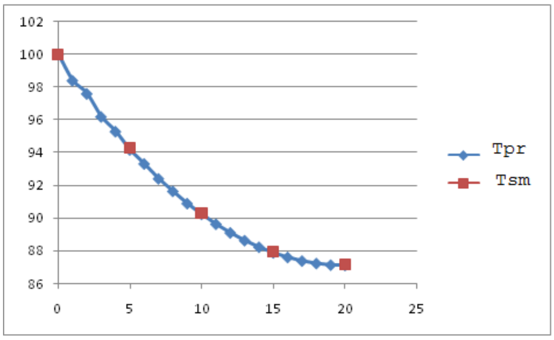

Fig 6: Comparison of field variable (T) with smooth solutions

Conclusions

1) In this work, A Dynamic nodal connectivity approach in Finite Element Method is developed using Bezier Basis Functions.
2) Using this method, the order of the degree can be increased without any change in nodal structure. High gradient regions can be solved without any re-meshing / adaptivity.
3) The above 4 test cases are considered for the validation of present method. It is observed from the test cases, that the results obtained from the current approach are in very good agreement either with the exact solution or the solution through the reference.

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