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Research Article

Influence of the uncertainties related to the Random Component of Rainfall Inflow in the Ouémé River Basin (Benin, West Africa)

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Abstract

Rainfall is very often considered as the driving force of hydrological models. If the rainfall changes, the model output (i.e. flow) is also expected to change. The objective of this paper is to study the impacts of the uncertainties related to the random component of rainfall inflow in the Ouémé river basin. The inflow process is considered as a sum of deterministic and random components. Hydrological systems are considered as non - linear dynamical systems which can be described by stochastic differential equations (SDE). The dynamics of the system is here derived from the Least Action Principle (LAP) considerations. Using data from Ouémé river basin (Benin, West Africa), the modelling of the random component using an ARMA model is investigated. The generalized Fokker – Planck equation (FPE) that corresponds to the SDE describing the river basin is derived in terms of the transition probability distribution and characteristic function of the noise generating process. This generalized FPE is used to examine the effects of different type of uncertainties related to the random component of rainfall inflow on the dynamics of river discharge. The form of the FPE is found to be particularly sensitive to the uncertainty properties of the inflowing rainfall.

Keywords: *ARMA model, Fokker - Planck equation (FPE), Least Action Principle (LAP), random component of inflow rainfall, stochastic differential equation (SDE), uncertainty.*

1. Introduction

The hydrological cycle has been greatly influenced by climate changes and human activities in the last decades (Solomon, et al, 2007). Today, evidence has been gained that the planet is warming up, largely as a result of human generated greenhouse gazes (IPCC, 2014a, b), (Kundzewicz, et al, 2014). Understanding the rainfall process is critical for the solution of several regional environmental problems of integrate water resources management (IWRM) at regional scales, with implications for agriculture, climate change and natural hazards such as floods and droughts (Abdul – Aziz, et al, 2013). The impact of rainfall errors on predicted flow has been highlighted by many authors, including Sun, et al, (2000), Kavetski, et al, (2006), Bárdossy and Das, (2008), and Moulin, et al, (2009). From a management perspective, inaccuracies in rainfall inputs directly compromise model predictions and hence robust decision - making on water and risk management options. The impact of input uncertainty on streamflow simulations can be quantified by error propagation, either by using conditional simulation or simply by stochastically perturbing the rainfall inputs. Conditional simulation involves simulating ensemble rainfall fields conditioned on the mean and error of spatial rainfall interpolations (Clark and Slater, 2006), (Götzinger and Bárdossy, 2008). Conditional simulation methods do not require many assumptions on rainfall errors (Clark and Slater, 2006), but can be time consuming to implement. Stochastic perturbation of rainfall inputs is therefore more common (Reichle, et al, 2002), (Carpenter and Georgakakos, 2004), (Crow and van Loon, 2006), (Pauwels and de Lannoy, 2006), (Komma, et al, 2008), (Pan, et al, 2008), (Turner, et al, 2008). In the stochastic perturbation approach it is common to perturb the model rainfall inputs based only on order of magnitude considerations. For example, Reichle, et al, (2002) used additive perturbations from a Gaussian distribution, with standard deviation equal to 50% of the rainfall total at each model time step. Given that uncertainty in hydrological simulations directly depends on adequate characterization of input error (Crow and van Loon, 2006), (Götzinger and Bárdossy, 2008). Therefore, detailed analysis of the observed error of rainfall inputs is a critical research priority.

To efficiently improve the development of integrated water resource management (IWRM) in the context of climate change and variability, an awareness of the stochastic structure of hydrologic processes is

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necessary for modelling water resources systems. In the present article, the inflow process representing all irregular variations of hydrometeorological processes is considered as a sum of deterministic and random components which embraces data uncertainties (e.g. measurement) and sample uncertainties (e.g. number of data). Adequate characterization of the uncertainties related to the random component of rainfall inflow is fundamental to success in rainfall - runoff modelling. No model, however well - founded in physical theory or empirically justified by past performance, can produce accurate runoff predictions if forced with inaccurate rainfall data (Beven, 2004). It is assumed that hydrological systems are non – linear dynamical systems which can be described by stochastic differential equations (SDE). Such equations arise when the elements which give rise to the representations of continuous deterministic dynamical system as ordinary differential equations (ODE) are considered subject to environmental fluctuations or noise. The theory is well developed and has found wide applications in most branches of sciences, including hydrology and water resources engineering ((Bodo, et al, 1987), (Konecny and Natchtnebel, 1995), (Hänggi, et al, 1995) and (Alamou, 2011). The new idea in this paper is that the dynamics of the system and the associated ODE are derived from the least action principle (LAP) designed to minimize uncertainties related to hydraulic transformation process and scaling law (physical models). Hence the deterministic hydrological model based on the least action principle (HyMoLAP) is fed by stochastic input. The main advantage of SDE is that it provides a physically transparent and mathematically tractable description of the stochastic dynamics, indicating how uncertainty in input precipitation and other environmental parameters (potential evapotranspiration, temperature) affect the uncertainty in model output. Using data from Ouémé river basin (Benin, West Africa) the random component is modelled with ARMA model to assess its stochastic properties. The generalized Fokker – Planck equation (FPE) that corresponds to the SDE describing the river basin is derived in terms of the transition probability distribution and characteristic function of the noise generating process. This generalized FPE is used to examine the effects of different type of uncertainties related to the random component of rainfall inflow on the dynamics of river discharge.

2. Methodology

2.1 Derivation of HyMoLAP

The proposed hydrological model based on the least action principle (HyMoLAP) uses the principle of minimum energy expenditure. This principle can be stated as follows: "Nature always follows the simplest way…. And the simplest way is the one which minimizes the energy expenditure of the nature" (Afouda, et al, 2004). The least action principle originally formulated in the 18th (by Maupertus, Euler, Lagrange, etc.) and generalized in the 20th century by Noether's theorem, has been of widespread application in fundamental physics (Arnold, 1974), (Doubrovine, et al, 1979) and is now conceived as a universal mathematical law of nature. It is herein considered that the global optimality principle postulated by Rodriguez - Iturbe and Rinaldo, (1997) for optimal channel networks is inherent to the self adjusting behaviour of a river basin endowed with optimal channel networks. This global optimality principle can be appropriately described by the LAP, whereby the entire pattern of motion is characterized using kinetic and potential energies, without specific reference to all the forces acting on or within the system. Although most of the classical applications of this principle assume that the kinetic and potential energies are respectively only functions of velocity and position, it is not necessarily the case in general. For hydrological systems, the discharge Q(X) which summarizes the interaction between water and the river basin medium at X can be considered as a generalized coordinate. The corresponding action is proposed in the form

$$\Lambda[Q] = \int L(Q, Q_X, X)dX \tag{1}$$

where L(.) is the Lagrangian, Q is the discharge, $Q_X = \frac{\partial Q}{\partial X}$ and X stands for time and space coordinate. For operational purposes the lumped version of the model is herein considered. Following Perrin, et al, (2003), the practical superiority of distributed or semi – distributed approaches over lumped ones for streamflow simulation has not been clearly demonstrated yet. The lumped version of HyMoLAP can be written as follows:

$$\frac{d}{dt}\left(\frac{dQ}{dt}\right) = \mu Q^{2\mu-1} \tag{2}$$

$$Q = K(\theta, t)q^b \tag{3}$$

where Q(t) is the discharge at the outlet of the river basin, q(t) is the cumulative rainfall and $K(\theta, t)$ is a time varying coefficient describing the interaction between the water flow and the river basin medium. μ and b are non – linearity coefficients. Equations (2) and (3) with the use of the following notations:

$\lambda = -\frac{d}{dt}(LogK)$, $Z = K\frac{dq^b}{dt}$, lead to equations (4) and (5)

$$\frac{dZ}{dt} = \psi(q,t) \tag{4}$$

$$\frac{dQ}{dt} = U(Q,\psi,\lambda) \tag{5}$$

with $U(Q, \psi, \lambda) = -\frac{\mu}{\lambda}Q^{2\mu-1} + \psi(q,t)$. ψ and U describe respectively the model input and structure. Assuming λ to be constant, a comparison has been made by Alamou, (2011) between the numerical value calculated from the model and the results from direct field measurement and thus confirm the physical meaning of this parameter. Clearly λ is the recession coefficient. Thus, the lumped properties of the river basin are described by λ while the hydrological properties are captured in the dynamical equations (4) and (5). Moreover equation (4) describes explicitly the production process (the action of the unsaturated zone which accounts for evaporation and evapotranspiration and divides the resulting rainfall event into two components: overland and underground) and equation (5) describes the transformation process (the process by which the amount of rainfall volumes for overland component and underground component are transformed into runoff). Here, equations (4) and (5) form the basic ODE describing the deterministic dynamics of the system. This model has been used successfully in rainfall - runoff modelling for the Bétérou catchment of the Ouémé river (Afouda and Alamou, 2010, Alamou, 2011).

2.2 ARMA model

The random component of the inflow process, for the time period (1961 - 2010), is calculated as defined by Lamb (1982):

$$\varepsilon_t = \frac{q_t - \overline{q}}{\sigma_q} \qquad (6)$$

where q_t is the spatial averaged rainfall of day t, $\bar{q}$ and σ_q represent respectively the mean and the standard deviation of this time series for the considered time period.

To assess the stochastic properties of the random component of the inflow process, the ARMA modelling approach is used. The models belonging to the ARMA (Autoregressive - moving average) family may be written as:

$$\varepsilon(t) = \sum_{i=1}^{p} \alpha_i \varepsilon(t-i) + \sum_{i=1}^{r} \beta_i w(t-i) + w(t) \qquad (7)$$

where { ε (t), t = 1 ,2, … } is the random component being modelled; p is the number of autoregressive (AR) parameters; α_i is the ith AR parameter; r is the number of moving average (MA) parameters; β_i is the ith MA parameter; { W (t), t =1, 2, … } is the residual series. The important assumption involved in such models is that W (t) is a sequence of white noise with zero mean and variance σ^2. The Box and Jenkins, (1976) three stage standard modelling procedure (identification, estimation and diagnostic checking) is used to develop time series models.

The first step is model identification: Identification of model consists of specifying the appropriate structure (AR, MA or ARMA) and order of model. Models can also be identified by looking at the plots of the autocorrelation function (ACF) and partial autocorrelation function (PACF). Thus making sure that the variables are stationary, identifying seasonality in the dependent series and using plots of the ACF and PACF of the dependent time series to decide which (if any) autoregressive or moving average component should be used in the model (Box and Jenkins, 1970).

The second step is to estimate the parameters of the model: Coefficients of the models can be estimated by maximum likelihood estimation or non - linear least - squares estimation methods. Estimation of parameters of MA and ARMA models usually requires a more complicated iteration procedure (Box and Jenkins, 1970), (Chatfield, 2004).

The third step is model checking: Two important elements of checking are to ensure that the residuals of the model are random, and to ensure that the estimated parameters are statistically significant. Usually the fitting process is guided by the principle of parsimony, by which the best model is the simplest possible model. Performing a Ljung - Box test or plotting autocorrelation and partial autocorrelation of the residuals are also helpful to identify misspecification (Anderson, 1976).

2.3 Derivation of the generalized FPE

The deterministic equations (4) and (5) can be transformed to a SDE by treating $\psi(q,t)$ and $U(Q,\psi,\lambda)$ as a random function that can be viewed as the sum of mean and a stochastic noise term. Afouda, et al, (2004) and Alamou, (2011) showed that the stochastic formulation of the system equations (4) and (5) can then be written in the vectorial form

$$dX(t) = U(X,t)dt + G(X,t)dW(t) \qquad (8)$$

However, in this paper, Let us use the scalar form of equation (8) in the form given by equation (9) for studying noise phenomena in hydrological systems

$$\frac{dQ}{dt} = U(Q,t) + G(Q,t)\varepsilon(t) \qquad (9)$$

where $U(Q,t) = -\frac{\mu}{\lambda}Q^{2\mu-1} + \psi(q,t)$ and $G(Q,t)$ are respectively the deterministic part and a multiplicative noise term. ε (t) is a noise resulting from a fluctuating environment.

This approach is especially effective if the noise that describes the action of the environment on the system can be represented as a time derivative, in the sense of generalized functions, of a stationary process with independent increments. In this case the solutions of equation (9) belong to the class of Markov processes whose properties are well known. The stationary process with independent increments and zero initial state constitutes a class of Lévy processes (Kato, 1999). For brevity, we call the Lévy process, whose derivative produces a given noise, the noise generating process. The statistical properties of solutions of equation (9) can be characterized by the transition probability distribution.

Let us derive the generalized FPE associated with equation (9) in terms of the transition probability and characteristic function of the noise generating process. Since the characteristic function is completely described by the transition probability distribution of the noise generating process, it is this distribution which ultimately determines the term in the generalized FPE that described the effect of the noise on the dynamics of the system. The starting point relies on the fact that the noise, ε (t), is the time derivative, in the sense of generalized functions, of the noise generating process η(t) (Gikhman and Skorokhod, 2004). According to this, the increment $\delta\eta(t) = \eta(t+\tau) - \eta(t)$ of η(t) is defined as the time integral,

$$\delta\eta(t) = \int_{t}^{t+\tau} dt'\varepsilon(t') \tag{10}$$

In the sense of convergence in distribution. Therefore the increment $\delta Q(t) = Q(t+\tau) - Q(t)$ of the discharge during a time interval τ (τ →0) can be written in the form

$$\delta Q(t) = U(Q,t) + G(Q,t)\delta\eta(t) \tag{11}$$

which defines the meaning of equation (9) in the Ito interpretation (Risken, 1989). For a fixed τ, the distribution of the increments $\delta\eta$(jτ) (j = 0, 1, 2 ,…) is completely described by the transition probability distribution p(η_{j+1}+τ |η_j), where η_{j+1} and η_j denote the possible values of η(jτ+τ) and η(jτ), respectively. Thus the statistical properties of solution of equation (9) can be characterized by p(η_{j+1}+τ|η_j) as well. Next for simplicity, we additionally assume that

$$p(\eta_{j+1},\tau \mid \eta_j) = p(\Delta\eta,\tau) \tag{12}$$

According to Denisov, et al, (2009), if one introduces the Fourier transform, $P_k(t)$, of the probability distribution function of the discharge, $P(Q,t)$, one obtains the following equation

$$\frac{\partial}{\partial t}P_k(t) = -F\left\{\frac{\partial}{\partial Q}U(Q,t)P(Q,t)\right\} + \int_{-\infty}^{\infty} dQ' e^{-ikQ'}\Phi_{kG(Q',t)}P(Q',t) \tag{13}$$

with $\Phi_k = \lim_{\tau\to 0}\frac{1}{\tau}[P_k(\tau) - 1]$ (14)

Since the transition probability distribution p(Δη, τ) is normalized, i.e., $P_0(\tau)$ = 1, equation (14) must satisfy the condition Φ_0 = 0. If k ≠ 0, then there exist three different cases, depending on how quickly $P_k(\tau) - 1$ tends to zero as τ →0.

- First, if $P_k(\tau) - 1$ =o(τ), then Φ_k = 0 and the noise has no effect on the system.
- Second, if $P_k(\tau) - 1$ tends to zero slower than τ, then |Φ_k | = ∞, i.e., the influence of the noise is so strong that the system relaxes instantaneously to the final state.
- Finally, the case we are interested in corresponds to $P_k(\tau) - 1$ =O(τ), i.e., 0 < |Φ_k | < ∞ and the noise acts on the system in a non – trivial way.

Let us apply the inverse Fourier transform, defined as

$$F^{-1}\{x_k\} \equiv x(Q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikQ} x_k, \tag{15}$$

to equation (13). Using equation (14), one obtains

$$F^{-1}\left\{e^{-ikQ'}\Phi_{kG(Q',t)}\right\} = \frac{1}{|G(Q',t)|}\Phi\left(\frac{Q-Q'}{G(Q',t)}\right), \tag{16}$$

where the function $\Phi(Q) = \lim_{\tau\to 0}[p(Q,\tau) - \delta(Q)]$ is a special characteristic of p(Δη,τ), for τ →0 that describes the influence of noise on the system. Therefore the desired generalized FPE that corresponds to the SDE (9) driven by multiplicative noise, which results from an arbitrary noise generating process, takes the form

$$\frac{\partial}{\partial t}P(Q,t) = -\frac{\partial}{\partial Q}U(Q,t)P(Q,t) + \int_{-\infty}^{\infty} dQ' \frac{P(Q',t)}{|G(Q',t|}\Phi\left(\frac{Q-Q'}{G(Q',t)}\right) \tag{17}$$

In accordance with the definition

$P(Q,t) = <\delta(Q - Q(t))>$, the solution of this equation must be normalized and satisfy the initial condition $P(Q,0) = \delta(Q)$.

To gain more insight into the connection between the generalized FPE and the properties of the noise, the characteristic function $S_k =< e^{-ik\eta(1)} >$ of the noise generating process η(t) at t =1 is introduced. With the formula

$$\eta(1) = \lim_{\tau\to 0}\sum_{j=0}^{[1/\tau]-1}\delta\eta(j\tau) \tag{18}$$

it can be rewritten as

$$S_k = \lim_{\tau\to 0}\left(P_k(\tau)\right)^{[1/\tau]} \tag{19}$$

where $P_k(\tau) =< e^{-ik\delta\eta(t)} >$ is the characteristic function of δη(t). Then, replacing $P_k(\tau)$ by $1 + \tau\Phi_k$ and taking into account that

$$\lim_{\xi \to 0} (1+\xi)^{1/\xi} = e, \quad (20)$$

we find $S_k = e^{\Phi_k}$, i.e., $\Phi_k = \ln S_k$. Thus, from equation (13) an alternative representation of the generalized FPE is obtained:

$$\frac{\partial}{\partial t} P(Q,t) = -\frac{\partial}{\partial Q} U(Q,t) P(Q,t) + F^{-1}\left\{ \int_{-\infty}^{\infty} dQ' e^{-ikQ'} P(Q',t) \ln S_{kG(Q',t)} \right\} \quad (21)$$

In the particular case of additive noise, where $G(Q,t) = 1$, equation (13) becomes

$$\frac{\partial}{\partial t} P_k(t) = -F\left\{ \frac{\partial}{\partial Q} U(Q,t) P(Q,t) \right\} + P_k(t) \Phi_k \quad (22)$$

and the generalized Fokker - Planck equation simplifies to the equation

$$\frac{\partial}{\partial t} P(Q,t) = -\frac{\partial}{\partial Q} U(Q,t) P(Q,t) + F^{-1}\{P_k(t) \ln S_k\} \quad (23)$$

The generalized Fokker - Planck equations (21) and (23) are derived here in terms of the characteristic function S_k of the noise generating process at t =1. The main advantage of this generalized Fokker - Planck equation is that it accounts for the noise action in a unified way, namely through the characteristic function of the noise generating process at dimensionless time t = 1. This provides an opportunity to study the effect of different noises on the same system.

3. Application to Ouémé river basin

3.1 The study area

In a scale of West Africa, Ouémé is a small coastal river that covers at Bonou, the most advanced station before the delta, a surface area of 49, 256 km² between 6.8 and 10.2° N of latitude and between 1.3 and 3.45° E of longitude (Figure 1). The Ouémé catchment covers two climatic zones: the Guinea savanna zone and the Soudanese savanna zone. The north of the catchment has a unimodal rainfall season (from mid - March to October) that peaks in August, whereas the south of the catchment exhibits a bimodal rainfall season (from March to July and from August to October) that peaks in June and September. This study is conducted, specifically, in the Bétérou, Savè and Bonou catchments of the Ouémé river. Meteorological data (daily rainfall data and daily potential evapotranspiration, calculated by the Penman formula) and daily discharge data were provided respectively by the Benin Meteorological Department, ASCENA (Agency for Air Navigation Safety in Africa and Madagascar) and the National Directorate of Water (DG - Eau). A total of twenty five rainfall stations were considered. In fact a homogeneity test of Kuskal - Wallis (Saporta, 1974) was conducted on every station of the study area, and this allowed to retain finally these twenty five stations. Moreover, the period 1961 - 2010 has been chosen as the study period (good compromise, taking into account the length of the data available in the different stations). Spatialized regional daily mean rainfall was obtained by kriging (Matheron, 1970) with an exponential variogram.

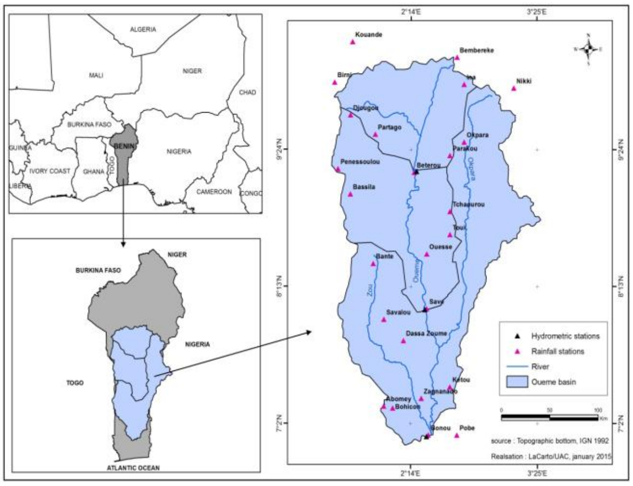


Fig.1 Location of the study area. The investigated sub catchments are Ouémé at Bétérou (10,475 km²), Ouémé at Savè (23,600 km²), Ouémé at Bonou (49,256 km²).

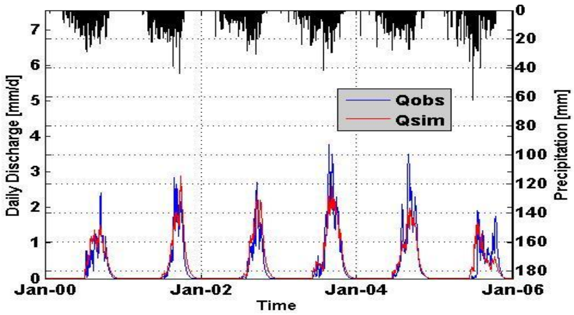
3.2 Simulation of discharge with HyMoLAP

Calibration of the model

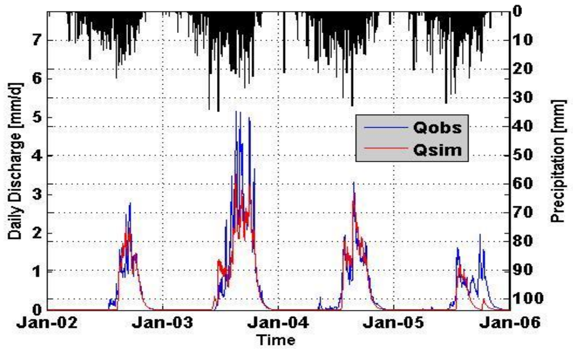
The model was calibrated for the time period 2000 - 2005, except for the Ouémé at Savè catchment where the calibration was performed over the time period 2002 - 2005, as 2000 and 2001 are missing. Figure 2 shows the results of the simulated hydrographs compared with the observed discharge for the calibration period. The difference between the observed and simulated results can be seen by a simple visual control and the numerical values for the coefficient of model efficiency (CE), the coefficient of determination (R^2) and the absolute percent bias (APB) as presented by Table 1. The uncertainties associated with the peaks are greater than those associated with low flow. The CE and R^2 are greater than 0.80 (Table 1), while an APB less than 40% was achieved.

Model validation

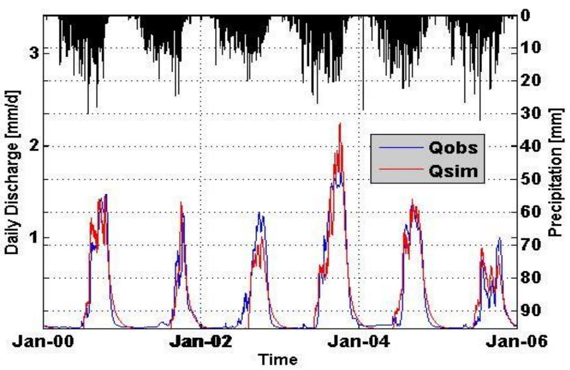
The model was applied with the same parameter set over the time period 2007 - 2008 for the investigated sub - catchments. The diagram of the model validation in the investigated sub - catchments in Figure 3 shows that the agreement of simulated and observed discharge is good. Comparable to the calibration period, the major differences between the measured and simulated hydrographs can be observed in the discharge peaks. The CE and R^2 are greater than 0.75 (Table 2), while an APB less than 40% was also achieved. These three coefficients used to evaluate the model prediction here are slightly lower values relative to the calibration period. These slightly lower values obtained after validation of the model can be explained by the deficiencies contained in the different climatic data in the model. However, these results indicate that the HyMoLAP is suitable for simulation of river discharge in these sub - catchments.



a) Oueme at Bétérou catchment



b) Oueme at Save catchment

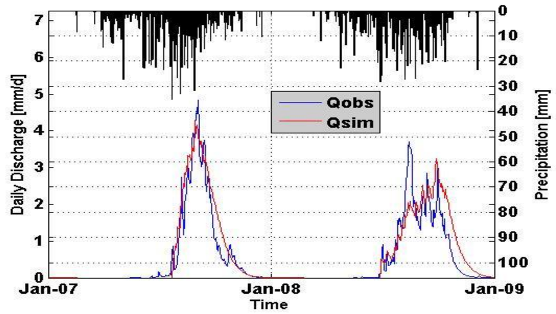


c) Oueme at Bonou catchment

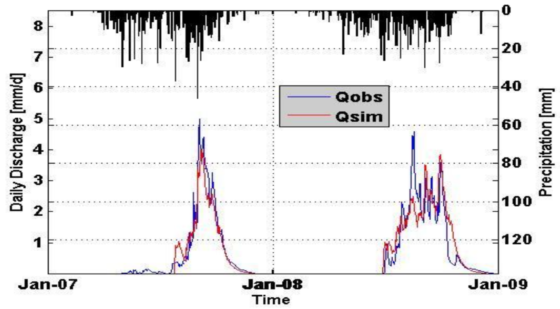
Fig.2 Simulated hydrographs compared with the observed discharge (calibration) for the investigated sub - catchments.

Table 1 Performance criteria of the HyMoLAP (Calibration) for the investigated sub – catchments

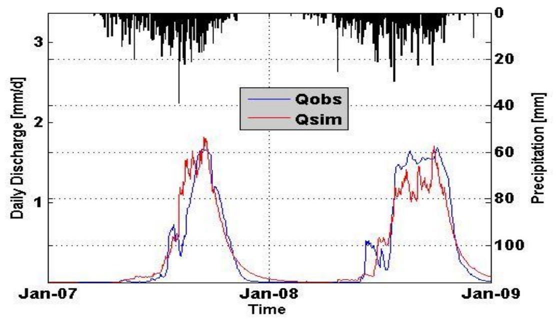
Sub - catchments	CE	R²	APB(%)
Bétérou	0.84	0.85	31.71
Save	0.82	0.86	33.81
Bonou	0.81	0.82	35.52



a) Oueme at Bétérou catchment



b) Oueme at Save catchment



c) Oueme at Bonou catchment

Fig.3 Simulated hydrographs compared with the observed discharge (validation) for the investigated sub - catchments.

Table 1 Performance criteria of the HyMoLAP (validation) for the investigated sub – catchments

Sub - catchments	CE	R²	APB(%)
Bétérou	0.78	0.81	31.71
Save	0.83	0.83	29.9
Bonou	0.76	0.78	39.52

However, there are still many sources of uncertainties not being taken into account by HyMoLAP (for instance the uncertainties which are related to the inflow process). The observed rainfall data that are often used for catchment studies are not areal rainfall because rainfall cannot be quantitatively measured in space with sufficient precision for catchment modelling. Usually, rainfall is only observed at some stations (point rainfall), located either inside or outside the study catchment. A lot of uncertainties due to measurement errors, spatial and temporal variability are therefore to address in hydrological modelling. This is done here through the modelling of the random component of the inflow process with ARMA model.

3.3 Modelling the random component with ARMA model

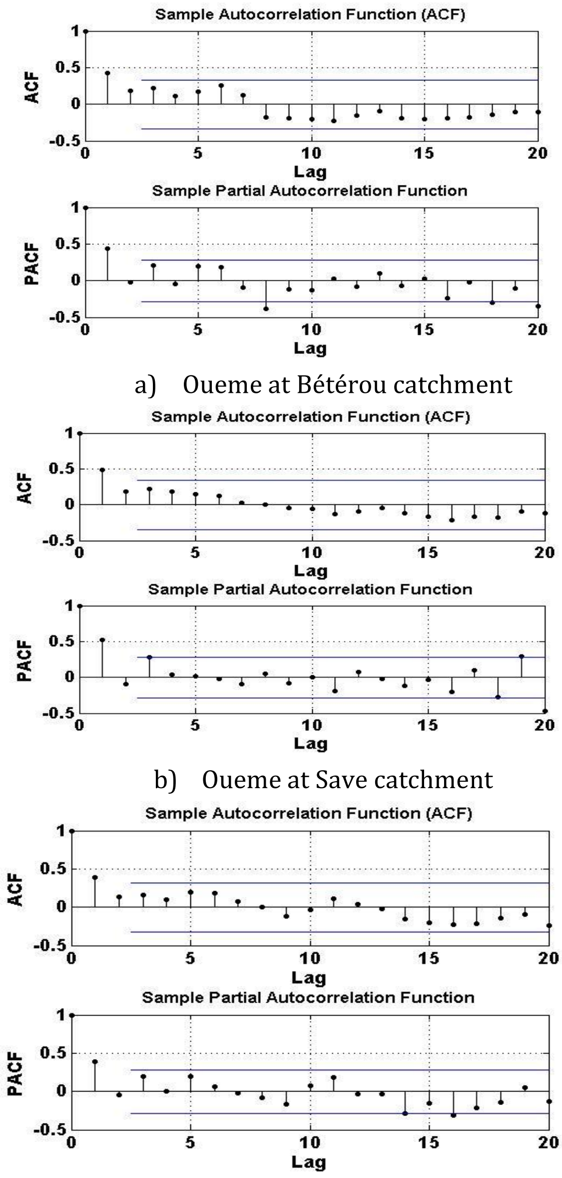
The autocorrelation function (ACF) and the partial autocorrelation function (PACF) were calculated for lags 0 to 20 using a 95% confidence level to identify the candidate models. This is because the first r terms in the ACF of a MA (r) process are non - zero and the remaining terms are all zeros. By the same way, the PACF indicates the order p of the AR process by the number of non - zero terms in the PACF. A visual inspection of the ACF graphs show that the autocorrelations of the random component in the investigated sub - catchments diminish fairly quickly. The series are therefore relatively stationary. The ACF and the PACF show both that the most significant value is observed at lag 1 (Figure 4).

From these observations, it was appropriate to try the ARMA (1, 1) model. However, for a better selection, the following models are investigated: ARMA (1, 1), ARMA (1, 2), ARMA (2, 1) and ARMA (2, 2) as they are mostly used in hydrology (Bacanli 2012).

Maximum likelihood estimation (MLE) is used to determine the efficient parameters. Table 3 gives the estimated parameters for the selected models to represent the random component.

However, it is possible to identify an ARMA model by using formal model selection criteria. The most widely used criteria are the Akaike information criterion (AIC), the Bayesian (Schwarz) information criterion (BIC or SIC) and the minimum residual variance Var (e). The best model is the one having the lowest AIC, BIC and Var (e). The results of the tested ARMA models using the goodness-of-fit criteria are summarized in Table 4. These results show that the best fitted model is ARMA (1, 1) model and confirm the findings about the use of ACF and PACF. The difference between the criteria does not seem important. A diagnostic checking is therefore conducted for all investigated models through careful analysis of residuals.

Let e_t be the sequence of residuals given by $e_t = \varepsilon_t - \widehat{\varepsilon_t}$. The basic assumption is that $\{e_t\}$ is a white noise (or that the series is uncorrelated). The selected models are validated by testing residuals for the significance of correlations. The results of the diagnostic checking revealed that ARMA (1, 1) model is effectively the best model and is therefore adequate for modelling the random component in the investigated sub-catchments. Indeed, a visual inspection of Figure 5 and Figure 6 shows that the ACF and the PACF for the residuals series do not display any significant correlations.



a) Oueme at Bétérou catchment

b) Oueme at Save catchment

c) Oueme at Bonou catchment

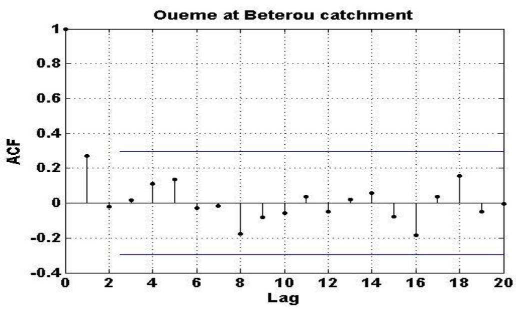
Fig.4 Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) for the random component for investigated sub - catchments. The horizontal lines give the threshold above which the correlation is significant at the level α =0.05.

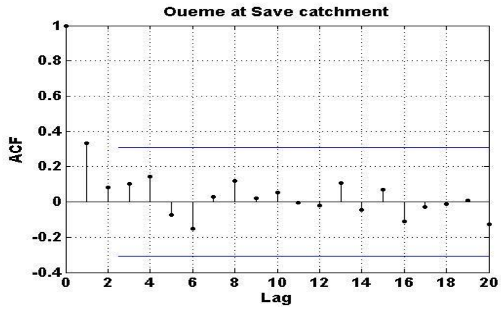
Table 3 Maximum Likelihood estimated Parameters of the tested ARMA models used to assess stochastic properties of the random component for the investigated sub – catchments.

Sub - catchments	ARMA models	α_1	α_2	β_1	β_2
Oueme at Bétérou	ARMA (1, 1)	0.017		0.5271	
	ARMA (1, 2)	0.793		-0.2916	-0.352
	ARMA(2, 1)	-0.0726	0.131	0.6338	
	ARMA(2, 2)	0.4796	0.2601	0.0225	-0.4437
Oueme at Save	ARMA (1, 1)	0.1905		0.4681	
	ARMA (1, 2)	0.8133		-0.2269	-0.3962
	ARMA(2, 1)	0.1966	0.0636	0.4755	
	ARMA(2, 2)	0.6697	0.193	-0.0665	-0.3645
Oueme at Bonou	ARMA (1, 1)	-0.0567		0.5568	
	ARMA (1, 2)	0.7961		-0.3517	-0.3351
	ARMA(2, 1)	-0.1816	0.1255	0.6932	
	ARMA(2, 2)	0.4884	0.3034	-0.05	-0.531

Table 4 Criteria of the ARMA models tested in assessing stochastic properties of the random component for the investigated sub - catchments.

Sub-catchments	ARMA models	AIC	BIC	Var(e)
Oueme at Bétérou	ARMA (1, 1)	134.0002	137.9027	1.6192
	ARMA (1, 2)	135.9176	141.7713	1.7679
	ARMA(2, 1)	135.8936	139.7473	2.4249
	ARMA (2, 2)	135.5522	143.3572	2.0992
Oueme at Save	ARMA (1, 1)	128.3026	132.2051	1.1814
	ARMA (1, 2)	129.5109	135.3647	1.8046
	ARMA(2, 1)	128.5602	134.414	1.5464
	ARMA (2, 2)	130.831	138.6358	1.7237
Oueme at Bonou	ARMA (1, 1)	135.9046	139.8071	1.4727
	ARMA (1, 2)	137.6411	143.4948	1.7155
	ARMA (2, 1)	136.1203	141.974	1.7743
	ARMA (2, 2)	137.2107	145.0156	2.5734





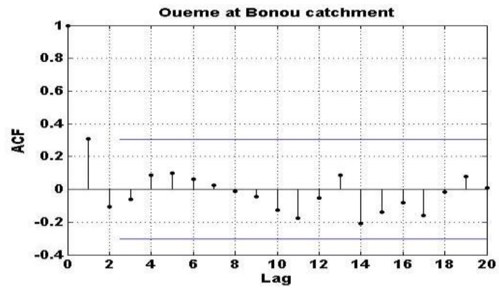


Fig.5 ACF plots with 95% confidence bands for the residuals series for the investigated sub - catchments.

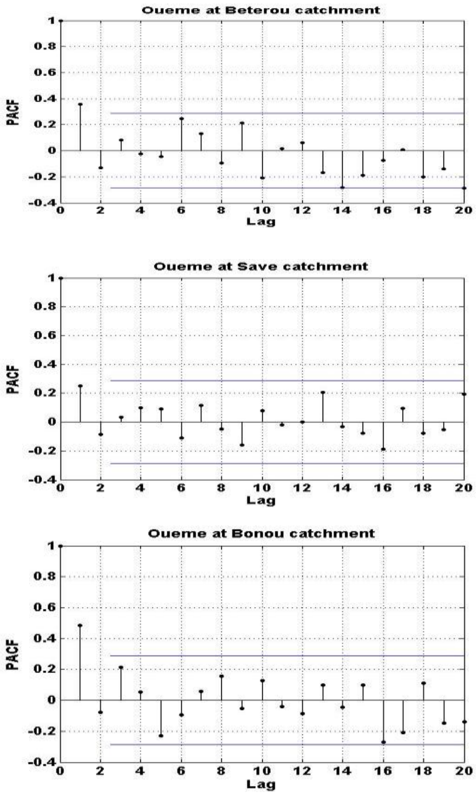


Fig.6 PACF plots with 95% confidence bands for the residuals series for the investigated sub - catchments.

In short, all the diagnostic checking conducted above show that the residuals series $\{e_t\}$ can be treated as white noise. Thus, ARMA (1, 1) model selected so far is adequate for modelling the random component in the investigated sub - catchments and confirms therefore the stochastic nature of the inflow process. These stochastic properties are based on the theory of stochastic differential equations and are used in the modelling of hydrological phenomena.

3.3 Effect of different specific noises on the dynamics of river discharge

a) Modelling the random component by Gaussian white noise

The transition probability distribution p(Δη,τ)for Gaussian white noise is given by equation (24)

$$p(\Delta\eta,\tau)=\frac{e^{-\Delta\eta^2/(4D\tau)}}{\sqrt{4\pi D\tau}} \qquad (24)$$

Accordingly, $P_k(k) = e^{-D\tau k^2}$, $\Phi_k = -Dk^2$, and $S_k = e^{-Dk^2}$. Then taking into account that

$$F^{-1}\{p_k(t)k^2\}=-\frac{\partial^2 P(Q,t)}{\partial Q^2} \qquad (25)$$

it can be found that in the case of additive Gaussian white noise, equation (23) reduces to the ordinary Fokker - Planck equation (Risken, 1989)

$$\frac{\partial P(Q,t)}{\partial t}=-\frac{\partial}{\partial Q}U(Q,t)P(Q,t)+ D\frac{\partial^2}{\partial Q^2}P(Q,t) \qquad (26)$$

If the Gaussian white noise is multiplicative, then $\Phi_{kG(Q',t)} = -Dk^2G(Q',t)$ and

$$F^{-1}\left\{\int_{-\infty}^{\infty}dQ'e^{-ikQ'}\Phi_{kG(Q',t)}P(Q',t)\right\}=D\frac{\partial^2}{\partial Q^2}G^2(Q',t)P(Q',t). \qquad (27)$$

Applying the inverse Fourier transform to equation (13) and using the above result, we obtain the ordinary Fokker - Planck equation (Risken, 1989)

$$\frac{\partial P(Q,t)}{\partial t}=-\frac{\partial}{\partial Q}U(Q,t)P(Q,t)+ D\frac{\partial^2}{\partial Q^2}G^2(Q,t)P(Q,t), \qquad (28)$$

b) *Modelling the random component by Poisson white noise*

In the specific case of Poisson noise, i.e., a random sequence of δ - pulses, ε(t) is defined as (Hänggi, 1978)

$$\varepsilon(t)=\sum_{i=1}^{n(t)}z_i\delta(t-t_i) \qquad (29)$$

Here $n(t)$ is a Poisson counting process with the probability $P(n(t) = n) = (\lambda t)^n e^{-\lambda t}/n!$ of $n \geq 0$ arrivals in the interval (0, t], λ is the rate of the process, t_i is the i^th (random) arrival time of this process, and z_i is the i^th independent random variable of zero mean distributed with the same probability q(z). It is assumed also that $\varepsilon(t) = 0$ if $n(t) = 0$. The noise generating process η(t) is a step - wise constant Markov process whose increments $\delta\eta(t) = \int_t^{t+\tau} dt'\varepsilon(t')$ are given by $\delta\eta(t) = 0$, if $\eta(t) = 0$ and $\delta\eta(t) = \sum_{i=1}^{\eta(\tau)} z_i$, if $\eta(t) \geq 1$.

The transition probability density p(Δη, τ) is given by

$$p(\Delta\eta,\tau)=(1-\lambda\tau)\delta(\Delta\eta)+\lambda\tau q_r(\Delta\eta) \qquad (30)$$

In this case, $S_k = e^{-\lambda(1-q_{r_k})}$, $q_r(Q) = F^{-1}\{q_{r_k}\}$ and $\delta(Q) = F^{-1}\{1\}$. Thus, equation (17) becomes

$$\frac{\partial P(Q,t)}{\partial t}=-\frac{\partial}{\partial Q}U(Q,t)P(Q,t)-\lambda P(Q,t)+ \lambda\int_{-\infty}^{\infty}dQ'\frac{P(Q',t)}{|G(Q',t|}q_r\left(\frac{Q-Q'}{G(Q',t)}\right). \qquad (31)$$

c) *Modelling the random component by compound noise*

The noise $\varepsilon(t) = \sum_{m=1}^{M} \varepsilon_m(t)$ is composed of a set of independent noises $\varepsilon_m(t)$. In this case the noise generating process can be written in the following form

$$\eta(t)=\lim_{\tau\to 0}\sum_{m=1}^{M}\sum_{j=0}^{[t/\tau]-1}\delta\eta_m(j\tau). \qquad (32)$$

Because of the statistical independence of the increments $\delta\eta_m(j\tau)$ of the partial generating processes $\eta_m(t)$, the characteristic function $S_k =< e^{-ik\eta(1)} >$ of η(1) is expressed through the characteristic functions $S_{m_k} =< e^{-ik\eta_m(1)} >$ of $\eta_m(1)$ as follows:

$S_k = \prod_{m=1}^{M} S_{m_k}$. Therefore, in the case of additive compound noise the generalized FPE (23) becomes

$$\frac{\partial P(Q,t)}{\partial t}=-\frac{\partial}{\partial Q}U(Q,t)P(Q,t)+\sum_{m=1}^{M}F^{-1}\{P_k(t)\ln S_{m_k}\}. \qquad (33)$$

In particular, if M = 2 and $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are Gaussian and Poisson white noises, respectively, equation (33) is then reduced to (Gardiner, 1990)

$$\frac{\partial P(Q,t)}{\partial t}=-\frac{\partial}{\partial Q}U(Q,t)P(Q,t)+D\frac{\partial^2}{\partial Q^2}P(Q,t) -\lambda P(Q,t)+\lambda\int_{-\infty}^{\infty}dQ'P(Q',t)q_r(Q-Q'). \qquad (34)$$

d) *Modelling the random component by Lévy stable noise*

We now assume that the random component of the inflow process is modelled by Lévy stable noise. The generalized central limit theorem (Gnedenko, et al, 1954) implies that for a wide class of properly scaled transition probability distribution p(Δη,τ), the characteristic function of the noise generating process

S_k corresponds to Lévy stable distributions, $S_k = S_k$ (α, β, γ, ρ). It is well known (Zolotarev, 1986) that S_k (α, β, γ, ρ) depends on four parameters: an index of stability α ∈ (0, 2], a skewness parameter β ∈[-1, 1], a scale parameter γ ∈ (0, ∞), and a location parameter ρ ∈ (-∞, ∞). Assuming, in accordance with the initial condition P(Q, 0) = δ(Q), that ρ = 0 and excluding from consideration the singular case when α = 1 and β ≠ 0 simultaneously (in this case $|\Phi_k| = \infty$), we obtain $S_k = S_k$ (α, β, γ) (Zolotarev, 1986), where

$$S_k(\alpha,\beta,\gamma)=\exp\left[-\gamma\,|k|^{\alpha}\left(1+i\beta\,\mathrm{sgn}(k)\tan\frac{\pi\alpha}{2}\right)\right] \quad (35)$$

In the following we assume for simplicity that the condition $G(Q',t) > 0$ holds for all Q' and t. In this case, $lnS_{kG(Q',t)} = G^{\alpha}(Q',t)lnS_k$(α, β, γ) and the generalized FPE (21) becomes

$$\frac{\partial P(Q,t)}{\partial t}=-\frac{\partial}{\partial Q}U(Q,t)P(Q,t)+F^{-1}\left\{G^{\alpha}(Q,t)P(Q,t)\right\} \quad (36)$$

where $G_k(t) = F\{G^{\alpha}(Q,t)P(Q,t)\}$.

By rewriting equation (36) in a form containing the Riemann - Liouville derivatives and using the characteristic function (35), we obtain the fractional FPE

$$\frac{\partial P(Q,t)}{\partial t}=-\frac{\partial}{\partial Q}U(Q,t)P(Q,t)-\frac{\gamma}{2\cos\left(\frac{\pi\alpha}{2}\right)}\left[(1+\beta)_{\infty}D_{+}^{\alpha}+(1-\beta)_{\infty}D_{-}^{\alpha}\right]G^{\alpha}(Q,t)P(Q,t) \quad (37)$$

Equation (37) reproduces all known forms of the fractional FPE that corresponds to the SDE (9) driven by Levy stable noise. It can be easily rewritten in a form containing the Riesz derivative defined as (Samko, et al, 1993)

$$\frac{\partial^{\alpha}}{\partial|Q|^{\alpha}}H(Q)=-F^{-1}\left\{|k|^{\alpha}\,h_k\right\} \quad (38)$$

With the help of this definition, and the relations

$$({}_{\infty}D_{+}^{\alpha}+{}_{\infty}D_{-})h(Q)=2\cos\frac{\pi\alpha}{2}F^{-1}\left\{|k|^{2}\,h_k\right\} \quad (39)$$

$$({}_{\infty}D_{+}^{\alpha}-{}_{\infty}D_{-})h(Q)=2\sin\frac{\pi\alpha}{2}\frac{\partial}{\partial Q}F^{-1}\left\{|k|^{\alpha-1}\,h_k\right\}, \quad (40)$$

we reduce equation (37) into

$$\frac{\partial}{\partial t}P(Q,t)=-\frac{\partial}{\partial Q}U(Q,t)P(Q,t)+\gamma\frac{\partial^{\alpha}}{\partial|Q|^{\alpha}}G^{\alpha}(Q,t)P(Q,t)+\gamma\beta\tan\frac{\pi\alpha}{2}\frac{\partial}{\partial Q}\frac{\partial^{\alpha-1}}{\partial|Q|^{\alpha-1}}G^{\alpha}(Q,t)P(Q,t). \quad (41)$$

It is obvious that for α = 2 this equation takes the form of the ordinary FPE (28) with D = γ.

4. Discussion

The degree of agreement between simulated and measured discharge that is observed over the investigated sub - catchments are similar to those obtained by Afouda and Alamou, (2010). The recession and low flow periods are well reproduced by HyMoLAP compared to the peaks. This is not the case for the simulation results of Götzinger, (2007) for the Ouémé at Bonou catchment. In his simulation, the recession does not match the observed data. The observed difference in the simulation of the recession period can be explained by the procedure used for the calculation of the parameters μ and λ of HyMoLAP, which considers a long period of drainage without rainfall input. In fact, the parameter λ is a recession coefficient. The deterministic modelling of the system dynamics by HyMoLAP reveals that the uncertainties associated with the peaks are greater than those associated with low flow. These high uncertainties associated with the peaks were also reported by Götzinger, (2007), who carried out a simulation with the HBV model for Ouémé at Bétérou catchment within the framework of the Rivertwin project. The fact that the discharge peaks are not well simulated can be attributed to data errors (Andréassian, et al, 2010), (Kuczera, et al, 2010). In fact, the improper representation of uncertainty is an intrinsic drawback of the deterministic hydrological models, since these do not include components that enable the preservation of the associated statistical characteristics of the observed data. With regard to the known drawback of deterministic hydrological models, the work of Efstratiadis, et al, (2014) confirmed that using an appropriate error model can effectively tackle the problem.

A remarkable feature of the derived FPE (28), (31), (34) and (41) is that they give an opportunity to examine the effects of different noises on the same system. Indeed, if one looks at these Fokker - Planck equations, it can be observed that the first terms on the right hand side are similar, whereas the remaining terms on this right hand side are different. In fact, it is these remaining terms which describe the effect of different specific noises on the dynamics of the system. As a consequence, the dynamics of the river discharge for each specific type of noise could not be the same. The above results clearly show how each specific type of noise can drastically modify the dynamics of the deterministic dynamical system. This means that inaccurate choice of noise models for dynamical equations will lead to poor decisions about the state estimates. Therefore, understanding the properties of the random components of the rainfall inflow on river water resources is crucial to successfully manage uncertainties in hydrological systems.

Conclusions

The main contribution of this paper is to investigate the influence of the uncertainties related to the random component of inflow process on the dynamics of river discharge. The achievement of this analysis stemmed from the combination of a two step modelling approach: (1) a deterministic modelling of the system dynamics by a hydrological model based on the least action principle (HyMoLAP), (2) the stochastic formulation of HyMoLAP in terms of stochastic differential equation (SDE). HyMoLAP is designed to minimize uncertainties related to hydraulic transformation process and scaling law, and thus characterized by a limited number of parameters (λ, μ) capable of physical interpretation, while its stochastic formulation helped to make use of the large body of Ito stochastic differential equation theory. The inflow process to hydrological modelling was considered as a sum of deterministic and random components.

Using data from Ouémé river basin (Benin), the stochastic properties of the random component of the inflowing precipitations have been investigated. This paper has demonstrated that the random component of the inflow process can be modelled with the ARMA (1, 1) model in the investigated sub - catchments. The dynamics of the system and the associated ODE are derived from the least action principle (LAP). The main advantage of the generalized Fokker - Planck equation is that it accounts for the noise action in a unified way, namely through the characteristic function of the noise generating process at dimensionless time t = 1. This provides an opportunity to study the effect of different specific noises on the same system. The form of the derived FPEs is found to be particularly sensitive to the uncertainty properties of the inflowing rainfall.

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