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Research Article

Mesh Less Method for the Solution of Forced Mechanical Oscillation Problems

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Abstract

B-spline collocation method is the implementation of B-spline as basis function in Collocation Method. It uses recursive formula of B-spline basis function for solving second order singular differential equations with Neumann's boundary conditions. Result of numerical examples show the accuracy and efficiency of this method. When the knot spacing is decreased or no.of knot is increased in the given domain the stability and efficiency are improved.

Keywords: *B-splines, Collocation Method, Machine, Mechanical oscillations and Motion equations.*

1. Introduction

Mathematical models in the form of differential and partial differential equations are used to represent various engineering problems in the fields such as Structural mechanics, Fluid flow, Heat transfer, Vibration analyses, etc. The solutions to these mathematical models can be Exact, Analytical or Approximate depending on the nature of these equations. When the Exact solution is not possible, numerical methods are needed to obtain approximate solutions. Many numerical techniques are evolved and has been used increasingly in last few years. Those numerical techniques include Finite Element Method [Ch.Sridhar Reddy *et al*, 2014], Finite Difference Method [R.K.Pandey *et al*, 2004], Kernel Space [Geng, F.Z. *et al*, 2007; Li. Z.Y. *et al*, 2012], B-spline collocation Method [Moshen. A *et al*, 2008], Predictor and Corrector Method [Abdalkaleg Hamad *et al*, 2014], and many more. The B-spline based Collocation Method is applied to evaluate boundary value problems including singular boundary value problems [Joan Goh *et al*, 2011].

The basis functions in B-spline curve are derived using equidistant knot space and for a particular degree only. A recursive formulation was given Carl.De boor [C.de Boor *et al*, 1982] for deriving these B-spline basis functions. If we use this formulation the evaluation of basis function can be generalized up to any degree. This basis function can be used in collocation method for uniform and non-uniform knot spacing.

In this paper the recursive formulation of B-spline basis functions [Hughes, T.J.R. *et al*, 2005, David F.Rogers *et al*, 2002] are discussed initially then the B-spline collocation method is discussed and formulated. The effectiveness and accuracy of this method is tested using the equation of motion in forced mechanical oscillations.

Considering the equation of motion in forced mechanical oscillations,

$$m\frac{d^2u}{dt^2}+k\frac{du}{dt}+nu=f(t) \qquad (1)$$

Where m the mass, $k\frac{du}{dt}$ the damping force resulting from the dashpot, $f(t)$ is external force and nu the restoring force resulting from the spring. Here, k and n are constants.

Let the approximation solution be

$$U^h(x)=\sum_{i=-2}^{n-1}C_iN_{i,p}(x) \qquad (2)$$

Where C_i are constants to be determined and $N_{i,p}(x)$ are B-Spline basis functions.

$U^h(x)$ is the approximate global solution to the exact solution $U(x)$ of the considered second order singular differential equation (1).

2. B-splines

2.1 Introduction

A spline is the mathematical representation of real world geometries. Schoenberg [David F.Rogers *et al*, 2002] was given first reference to the word B-spline in the middle of 19th century. He described it as a smooth piecewise

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polynomial curve. Carl De Boor's [C.de Boor *et al*, 1982] publication on faster and numerically stable algorithms for the calculation of spline interpolation functions in 1972, marked the beginning of the use of splines in automobile modelling. Form mathematical point of view, a curve generated by using the vertices of a defining polygon and the curve is dependent on some interpolation or approximation scheme between the curve and polygon.

This scheme is provided by the choice of B-Spline basis functions. B-spline basis is generally has non global behavior due to the property that each vertex of B-spline B_i is associated with a unique basis function. Thus, each vertex affects the shape of a curve only over a range of parameter values where its associated basis function and hence the degree of the resulting curve to be changed without changing the number of defining polygon vertices.

Any curve can be represented as a parametric curve, i.e. the coordinate *x* is represented as a function of a parameter *t*. A parametric B-spline curve can be defined by

$$p(x)=\sum_{i=0}^{n} B_i N_{i.p}(t) \quad (3)$$

Where $x_{\min} \le x < x_{\max}$, $2 \le p < n+1$

Where the B_i are the position vector of the n+1 defining polygon vertices, *p* is the degree and the $N_{i,k}(x)$ are the normalized B-spline basis functions.

i) If *p*=0

$$N_{i,p}(x)=\begin{cases}1 & \text{if} \quad x_i \le x < x_{i+1}\\ 0 & \text{otherwise}\end{cases}$$

ii) If *p*≥1

$$N_{i,p}(t)=\frac{(x-x_i)N_{i,p-1}(x)}{x_{i+k}-x_i}+\frac{(x_{i+p+1}-x)N_{i+1,p-1}(x)}{x_{i+p+1}-x_{i+1}} \quad (4)$$

The values of x_i are elements of knot vector satisfying the relation $x_i \le x_{i+1}$. The parameter x varies from x_{min} to x_{max} along the curve p(x).

The Partition of unity property ensures that the relationship between the curve and its defining control points is invariant under affine transformations. The sum of the B-spline basis functions is 1 for any parameter value x. The Positivity property guarantees that the curve segment lies completely within the convex hull of P_i. The Local support property indicates that each segment of a B-Spline curve is influenced by only *p* control points or each control point affects only *p* curve segments.

2.2 B-spline Derivatives

Since the goal of B-spline collocation Method is to compute approximation solution for differential Equations, so the derivatives of the B-spline basis functions needs to be calculated. B-spline basis functions are defined recursive formula. Equations 5 and 6 are the first and second derivatives of B-spline curve of order 2.

The first derivative of B-spline curve is

$$p^{'}(x)=\sum_{i=0}^{n} B_i N^{'}{}_{i.p}(t)$$

Where

$$N'_{i,p}(x)=\frac{x-x_i}{x_{i+p}-x_i}N'_{i,p-1}(x)-\frac{N_{i,p-1}(x)}{x_{i+p}-x_i}+\frac{x_{i+p+1}-x}{x_{i+p+1}-x_{i+1}}N_{i+1,p-1}(x)+\frac{N_{i+1,p-1}(x)}{x_{i+p+1}-x_{i+1}} \quad (5)$$

The second derivative of B-spline curve is

$$p^{''}(x)=\sum_{i=0}^{n} B_i N^{''}{}_{i.p}(t)$$

Where

$$N''_{i,p}(x)=2\frac{N'_{i,p-1}(x)}{x_{i+p}-x_i}-2\frac{N'_{i+1,p-1}(x)}{x_{i+p+1}-x_{i+1}} \quad (6)$$

From above equations 5 and 6, the basis functions are expressed as recursively in terms of previous degree basis function i.e. the p^{th} degree basis functions is the combination of ratios of knot values and (p-1) degree basis function. Again $(p-1)^{th}$ degree basis function is defined as the combination ratios of knot values and (p-2) degree basis function. In a similar way every B-Spline basis function of degree up to (p-(p-2)) is expressed as the combination of the ratios of knot values and its previous B-Spline basis functions.

B-Spline basis of second degree over uniform knot vector is shown graphically below in figure 1 and 2.

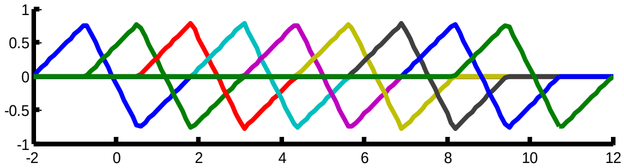

Figure 1: First derivative of second degree B-spline basis function with uniform Knot vector X= {-2,-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

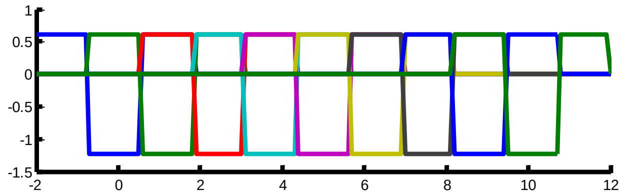

Figure 2: Second derivative of second degree B-spline basis function with uniform Knot vector X= {-2,-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

2.3 B-spline collocation Method

Collocation method is the most widely used approximation method for solving motion equation of forced mechanical oscillations. In normal collocation method we use polynomials whereas in B-Spline collocation method we use B-Spline basis functions. The B-spline approximations together with Collocation Method produces an economical alternative for solving many engineering problems which are in the form of differential equations. The selection of nodes or collocation points is arbitrary. The main aim of this paper is to analyze the efficiency of the B-spline based collocation method for such problems with sufficient accuracy.

2.3.1 Formulation of B-spline Collocation Method:

As stated earlier B-spline functions are used as the basis in B-spline collocation method whereas the base functions which are used in normal collocation method are the polynomials vanishes at the nodes. Let [a, b] be the domain of the governing equation (1) for modelling of forced mechanical oscillations and the domain is partitioned as $X=\{a=x_0, x_1, x_2, \ldots x_{n-1}, x_n=b\}$ with equal length $h=\frac{b-a}{n}$ for n sub domains. The x_i*'s* are known as nodes, these nodes are treated as knot values in B-Spline collocation method. Assumed that the residue is equal to zero in order to determine unknowns C_i's in equation (2), i.e., Residue which is obtained by substituting equation (2) in equation (1) is made equal to zero at collocation points or nodes in the given domain to determine unknowns in (2). Two extra knot values are taken into consideration both sides of the domain of problem when evaluating the second degree B-Spline basis functions at the nodes. These extra knots are taken to satisfy the partition of unity property and to get accurate B-spline basis functions.

First derivative of approximation function (2) is

$$\frac{dU^h(x)}{dx}=\sum_{i=-2}^{n-1}C_i N'_{i,p}(x) \tag{7}$$

Second derivative of approximation function (2) is

$$\frac{d^2U^h}{dx^2}=\sum_{i=-2}^{n-1}C_i N''_{i,p}(x) \tag{8}$$

Substituting, the approximate solution (2) in (1) we have,

$$m\frac{d^2U^h}{dx^2}+\frac{dU^h}{dx}+nU^h = f(x) \tag{9}$$

Substituting the Approximation function and its derivatives (2), (7) and (8) in the equation (1), we have

$$\sum_{i=-2}^{n-1}C_i N''_{i,p}(x)+k_1P(x)\sum_{i=-2}^{n-1}C_i N'_{i,p}(x)+k_2Q(x)\sum_{i=-2}^{n-1}C_i N_{i,p}(x)= f(x) \tag{10}$$

i.e.

$$\begin{aligned}&[C_{-2}N''_{-2,p}(x)+C_{-1}N''_{-1,p}(x)+C_0N''_{0,p}(x)+\ldots\ldots+C_{n-1}N''_{n-1,p}(x)]+\\&k_1P(x)[C_{-2}N'_{-2,p}(x)+C_{-1}N'_{-1,p}(x)+C_0N'_{0,p}(x)+\ldots\ldots+C_{n-1}N'_{n-1,p}(x)]+\\&k_2Q(x)[C_{-2}N_{-2,p}(x)+C_{-1}N_{-1,p}(x)+C_0N_{0,p}(x)+\ldots\ldots+C_{n-1}N_{n-1,p}(x)]= f(x)\end{aligned}$$

i.e.

$$\begin{aligned}&[N''_{-2,p}(x)+k_1P(x)N'_{-2,p}(x)+k_2Q(x)N_{-2,p}(x)]C_{-2}+\\&[N''_{-1,p}(x)+k_1P(x)N'_{-1,p}(x)+k_2Q(x)N_{-1,p}(x)]C_{-1}+\\&[N''_{0,p}(x)+k_1P(x)N'_{0,p}(x)+k_2Q(x)N_{0,p}(x)]C_0\ldots\ldots\\&\ldots\\&\ldots\ldots+[N''_{n-1,p}(x)+k_1P(x)N'_{n-1,p}(x)+k_2Q(x)N_{n-1,p}(x)]C_{n-1}= f(x)\end{aligned} \tag{11}$$

The matrix representation of system (11) at x_i's, i=0, 1, 2,….n-1 gives the system of (n-1)×(n+1) equations in which (n+1) arbitrary constants are involved.

Now the Matrix (13) can be written as

$$\begin{bmatrix} N_{-2}(0) & N_{-1}(0) & N_0(0)\ldots.. & N_{n-1}(0)\\ N_{-2}(1) & N_{-1}(1) & N_0(1)\ldots.. & N_{n-1}(1)\\ N_{-2}(2) & N_{-1}(2) & N_0(2)\ldots.. & N_{n-1}(2)\\ . & . & . & .\\ . & . & . & .\\ . & . & . & .\\ N_{-2}(n-1) & N_{-1}(n-1) & N_0(n-1)\ldots.. & N_{n-1}(n-1)\end{bmatrix}\begin{bmatrix}C_{-2}\\C_{-1}\\C_0\\.\\.\\.\\C_{n-1}\end{bmatrix}=\begin{bmatrix}f(0)\\f(1)\\f(2)\\.\\.\\.\\f(n-1)\end{bmatrix} \tag{12}$$

Applying boundary conditions to approximate solution, we have

$$\sum_{i=-2}^{n-1}C_i N_{i,p}(a)=d_1$$

$$\sum_{i=-2}^{n-1}C_i N_{i,p}(b)=d_2 \tag{13}$$

A (n+1)×(n+1) square is obtained from by using the equations (12), (13), we have

$$\begin{bmatrix} N_{-2}(a) & N_{-1}(a) & N_0(a)\ldots. & N_{n-1}(a)\\ N_{-2}(0) & N_{-1}(0) & N_0(0)\ldots. & N_{n-1}(0)\\ N_{-2}(1) & N_{-1}(1) & N_0(1)\ldots. & N_{n-1}(1)\\ N_{-2}(2) & N_{-1}(2) & N_0(2)\ldots. & N_{n-1}(2)\\ . & . & ..\ldots & .\\ . & . & ..\ldots & .\\ . & . & ..\ldots & .\\ N_{-2}(n-1) & N_{-1}(n-1) & N_0(n-1)\ldots. & N_{n-1}(n-1)\\ N_{-2}(b) & N_{-1}(b) & N_0(b)\ldots. & N_{n-1}(b)\end{bmatrix}\begin{bmatrix}C_{-2}\\C_{-1}\\C_0\\.\\.\\.\\.\\.\\C_{n-1}\end{bmatrix}=\begin{bmatrix}d_1\\f(0)\\f(1)\\f(2)\\.\\.\\.\\f(n-1)\\d_2\end{bmatrix} \tag{14}$$

It is in the form of $[N][C]=[R]$ (15)

The local support of B-spline basis functions causes to have diagonally dominated for the matrix [N]. So that the system of equations (15) is easily solved for the arbitrary constants C_i's.

We have

$$[C]=[R][N]^{-1} \tag{16}$$

The approximate solution becomes as known solution is obtained. Now the final approximation solution by substituting these constants in equation (2). This approximate solution is used to evaluate the field variable at each node (Collocation point) in the considered domain. The exact solution is also evaluated at these points and result values are compared with each other to find out the accuracy of the B-spline Collocation Method.

2.4 Numerical Examples

A numerical example is considered for testing the efficiency and convergence of the B-spline Collocation Method

2.4.1 Case I

Considering the motion equation of forced mechanical oscillation, when the mass is displaced a distance x initially and released from rest. Then at t = 0, *u=0* and du/dt=0 and mass is subject to a force *f(t)*.

$$\frac{d^2u}{dx^2}+u=x^2 \quad \text{for } 0 < x < 1 \tag{17}$$

With boundary conditions u(0) = u`(0) = 0.

The exact solution of above equation is u(x)

u(x)= $2\cos(x)+x^2-2$

Taking number of intermittent segments (or sub domains) as 20 (i.e. n=20), degree of B-Spline curve as 3 (i.e. p=3).

X= {a= X_0 =0, X_1, X_2, …. X_{n-1}, X_n=b} with equal length $h=\frac{b-a}{n}=\frac{1-0}{20}=0.05$ and knot vector having 24 elements or knot values. Now the above equation can be modified as

$$U^h(x)=\sum_{i=-2}^{19}C_iN_{i,2}(x) \tag{18}$$

Substituting the approximation function in governing equation we have

$$\sum_{i=-2}^{19}C_iN''_{i,2}(x)+\sum_{i=-2}^{19}C_iN_{i,2}(x)=x^2 \tag{19}$$

Knot vector is X= {-0.10-0.05 0 0.05 0.10 0.15 0.20 0.250 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1 1.05 1.10}

By solving the set of equations we get the constants C_i where i=0, 1, 2, 3 …19. And by substituting these constant values in approximation solution equation then we get the final solution for the given problem.

Now the final approximation solution is evaluated at each node (Collocation point) i.e. i=0, 1, 2, 3 …19 and the values field variable u(x) at each node are calculated. The exact solution also evaluated at these points and result values of field variable u(x) are compared with each other to find out the accuracy of the B-spline Collocation Method and shown in table below.

Table 1: Comparison of field variable *u*(*x*) of BSCCM solution with exact solution

Node	Exact Sol	BSCCM Sol
0	0	0
0.05	5.21E-07	-5.47E-16
0.1	8.33E-06	3.12E-06
0.15	4.22E-05	2.19E-05
0.2	0.0001	8.12E-05
0.25	0.0003	0.0002
0.3	0.0006	0.0004
0.35	0.0012	0.0009
0.4	0.0021	0.0016
0.45	0.0033	0.0027
0.5	0.0051	0.0042
0.55	0.0075	0.0063
0.6	0.0106	0.009
0.65	0.0146	0.0126
0.7	0.0196	0.0171
0.75	0.0258	0.0227
0.8	0.0334	0.0296
0.85	0.0424	0.0379
0.9	0.0532	0.0478
0.95	0.0658	0.0596
1	0.0806	0.0775

As we decrease knot space i.e. mesh size the relative error is decreased and that gives the convergence of the method. Maximum modulus error is constantly decreasing as the number of nodes are increased. This shows that B-spline Collocation Method gives is the best method to approximate the exact solution.

The following figure compares the approximate solution calculated by B-spline Collocation Method with exact solution calculated at the same collocation points.

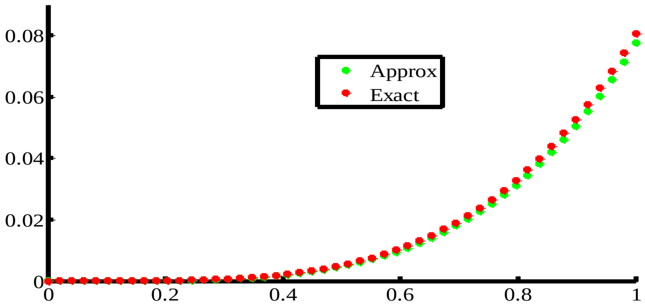

Figure 3: Comparison of Approximate solution (u) with exact solutions with knot space *h*=0.05

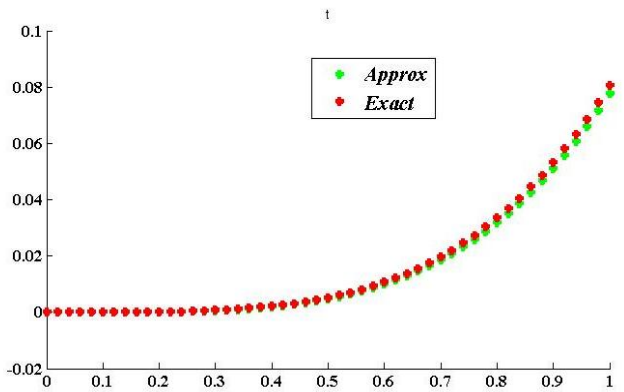

Figure 4: Comparison of Approximate solution (u) with exact solutions with knot space *h*=0.02

Conclusions

The B-Spline basis functions are defined recursively and incorporated in the collocation method. This method is applied for modelling of forced mechanical oscillations. The accuracy of the present method is illustrated by considering a numeric example. The B-spline collocation method solution is compared with exact solution and found to be in best fit approximation.

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